

How Can the Mismatched Cramér-Rao Bound be Useful in Satellite Positioning?

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Abstract—This paper investigates the computation of a modified mismatched Cramér-Rao Bound (CRB) for time-of-flight estimation in Global Navigation Satellite Systems (GNSS). The main issue is the mismatch between the nominal ranging signal and the actual received signal due to various systematic errors, including distortion and multipath propagation. The traditional CRB computed on the nominal signal is clearly inadequate under such conditions, leading to the introduction of a Mismatched CRB (MMCRB) that reflects the actual receiver's operating environment. The paper further introduces a Modified MMCRB (M^3 CRB) that gets rid of the dependence of the bound on the specific ranging code. This work ultimately aims to enhance GNSS payload design using these new performance metrics.

Index Terms—GNSS, ranging, modified mismatched CRB, payload design.

I. INTRODUCTION

The derivation of the user receiver position in a GNSS system is still largely based on indirect ranging measurements. The receiver estimates the positions of the diverse *anchors* (i.e., the satellites) in the positioning problem through time-of-flight estimation of the ranging signals transmitted by the different satellites [1]. Amidst the many systematic error sources that introduce an unknown error into this measurement (ephemeris, satellite clock, ionosphere propagation, etc.), we are interested here in the *mismatch* between the (nominal) waveform expected by the receiver, according to Signal-in-Space (SIS) specifications, and the actual received signal. The reasons of this mismatch may be a few: linear or nonlinear distortion introduced by the on-board Navigation Signals Generation Unit (NSGU), multipath propagation, I/Q receiver imbalance, etc. [5].

Very often, the user receiver has *not* the possibility to model, estimate, and/or compensate the unknown distortion on the received signal. In such conditions, computing the conventional CRB [2] for time-of-flight estimation in Additive White Gaussian Noise (AWGN) is not fair – it can never be attained because of the difference (the *mismatch*) between the nominal signal shape, and what is actually available for measurement and estimation. Therefore, the correct Key Performance Indicator (KPI) to assess ranging accuracy turns out to be the *Mismatched CRB* (MMCRB, *aka* Mis-Specified CRB) [3], that captures the impossibility of the receiver to

attain the nominal performance, since the nominal signal is not observable in its un-distorted form.

A further issue related to GNSS signals parameter estimation, and in particular computation of the estimation CRB, is the dependence of the estimate/bound on the *specific ranging code* embedded into the signal under estimation. Within the same system (typically GPS and GALILEO), the different pilot signals coming from different satellites on the same frequency share the very same signal format, apart from the specific ranging code chip sequence [1]. In reality, all such ranging codes are long pseudo-random binary sequences, so that the dependence of the CRB on the code is vanishingly small, the “common” CRB of all pilot signals in the same system and with the same format is virtually independent of the ranging code, and is very well approximated by the so-called *Modified CRB* [4], much simpler to compute than the ranging-code-dependent (true) CRB.

In this paper, we revise the derivation of the MMCRB for time-of-flight (i.e., time delay) estimation of a ranging signal and, following the approach above, we also introduce the Modified MMCRB (M^3 CRB) to get rid of the dependence of the bound on the particular ranging code. We also give preliminary examples of the application of such new bound in a condition of mild-to-severe signal distortion, and we finally propose to use it as the Key Performance Indicator (KPI) in signal and payload design for GNSS.

II. THE MMCRB FOR DELAY ESTIMATION

Assuming coherent baseband processing (a simplified assumption to be released later on), the *ideal* received signal on the observation window $0 \leq t < T_{obs}$ is:

$$r_i(t) = x(t - \tau) + w(t) \quad , \quad 0 \leq t < T_{obs} \quad (1)$$

where $w(t)$ is Additive White Gaussian Noise (AWGN) with power spectral density $S_w(f) = N_0/2$, and where

$$\int_0^{T_{obs}} |x(t)|^2 dt = E_x \quad (2)$$

is the signal energy within the observation time. What the receiver actually gets is in general different, because of unpredictable linear/nonlinear distortion introduced by the NSGU on-board the satellite and/or by the propagation channel. The *actual* received signal is

$$r_a(t) = y(t) + w(t) \quad , \quad 0 \leq t < T_{obs} \quad (3)$$

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where $y(t)$ is a (hopefully slightly) distorted version of $x(t - \tau)$. The distortion creates a *mismatch* between the ideal (1) and the actual (3) signal models, so that (3) turns out to be *mis-specified*. Parameter estimation on the received signal obeys therefore a different bound, the so-called *Mis-Matched Cramer Rao Bound* (MMCRB) [3] that represents the actual best performance the receiver can attain in terms of estimation Mean-Square Error.

To compute more easily the MMCRB, we start from the time-discrete formulation of (3), obtained after sampling with a frequency f_s :

$$r_a[n] = y[n] + z[n] \quad , \quad n = 0, 1, \dots, N-1 \quad (4)$$

where $r_a[n] = r(nT_s)$, $y[n] = y(nT_s)$, and $z[n]$ is zero-mean AWGN with variance $\sigma_w^2 = \sigma_Q^2 = N_0 f_s / 2 \triangleq \sigma_z^2$. We can also assume that $f_s T_{obs} = N$, N integer, so that we have (exactly) N samples $r_a[n]$, $n = 0, 1, \dots, N-1$ within our observation time. In the following, we will also use the notation $\mathbf{r}_a = [r_a[0], r_a[1], \dots, r_a[N-1]]$ and similarly \mathbf{r}_i , \mathbf{y} , $\mathbf{x}(\tau)$ to indicate the N -dimensional arrays collecting the samples of the respective signals. Conditioned on the value of the parameter under estimation τ , both the mismatched and the actual signal models bear Gaussian statistics, the mismatch lying in the different (conditional) expected value.

A first conceptual issue related to the computation of the bound regards the notion of the “true” value of the parameter under estimation. In the ideal signal model (1), the true value is unambiguous: it is τ . When receiving the actual signal (3), this notion is lost since the “delay” of $y(t)$ is not clearly defined. As a first step, we define the *pseudo-true* value τ_0 of the parameter under estimation τ as that value that minimizes the *Kullback-Leibler divergence* $D(f_{\mathbf{r}_a}(\mathbf{r}) || f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau}))$ of the probability density function (pdf) of the *actual* received signal vector $f_{\mathbf{r}_a}(\mathbf{r})$ wrt the pdf of the *ideal* signal vector, conditioned on a particular trial value $\tilde{\tau}$ of the delay:

$$\begin{aligned} \tau_0 &\triangleq \underset{\tilde{\tau}}{\operatorname{argmin}} D(f_{\mathbf{r}_a}(\mathbf{r}) || f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau})) \\ &= \underset{\tilde{\tau}}{\operatorname{argmin}} \int_{\mathbf{r}} \log_2 \left(\frac{f_{\mathbf{r}_a}(\mathbf{r})}{f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau})} \right) f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau}) d\mathbf{r} \end{aligned} \quad (5)$$

This parameter has the same role in the actual received signal \mathbf{r}_a as τ has in the ideal signal model \mathbf{r}_i . Computation of $D(f_{\mathbf{r}_a}(\mathbf{r}) || f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau}))$ is relatively simple because the two pdf's are both Gaussian multivariate with the same covariance matrix $\mathbf{C}_{\mathbf{r}_i} = \mathbf{C}_{\mathbf{r}_a} = \sigma_z^2 \mathbf{I}$, i.e.,

$$\begin{aligned} f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau}) &= \frac{1}{(2\pi\sigma_z^2)^{N/2}} \exp \left\{ -\frac{\|\mathbf{r} - \mathbf{x}(\tilde{\tau})\|^2}{2\sigma_z^2} \right\} \\ f_{\mathbf{r}_a}(\mathbf{r}) &= \frac{1}{(2\pi\sigma_z^2)^{N/2}} \exp \left\{ -\frac{\|\mathbf{r} - \mathbf{y}\|^2}{2\sigma_z^2} \right\} \end{aligned} \quad (6)$$

The result is

$$\begin{aligned} D(f_{\mathbf{r}_a}(\mathbf{r}) || f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau})) \\ = \frac{1}{2\sigma_z^2} \sum_{n=0}^{N-1} |y[n] - x[n; \tilde{\tau}]|^2 = \frac{\|\mathbf{y} - \mathbf{x}(\tilde{\tau})\|^2}{2\sigma_z^2} \end{aligned} \quad (7)$$

When the observation time is large, $\sum |x[n; \tilde{\tau}]|^2$ does not depend on $\tilde{\tau}$, therefore the value that minimizes D in (7) is directly found by maximizing the cross correlation between the received signal and the ideal signal model, just like in Maximum-Likelihood delay estimation:

$$\tau_0 = \underset{\tilde{\tau}}{\operatorname{argmax}} \sum_{n=0}^{N-1} y[n] x[n; \tilde{\tau}] \quad (8)$$

Alternatively, to find the minimum, we can differentiate (7) wrt to $\tilde{\tau}$, obtaining the necessary condition for minimization to hold:

$$\sum_{n=0}^{N-1} (y[n] - x[n; \tilde{\tau}_0]) \dot{x}[n; \tilde{\tau}_0] = 0 \quad (9)$$

Once τ_0 is found, the MMCRB is as follows [3]:

$$\sigma_{\tilde{\tau}}^2 \geq \frac{b(\tau_0)}{a^2(\tau_0)} \quad (10)$$

where

$$\begin{aligned} a(\tau_0) &\triangleq E_{f_{\mathbf{r}_a}} \left\{ \frac{\partial^2 \ln f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau})}{\partial \tilde{\tau}^2} \right\} \Big|_{\tilde{\tau}=\tau_0} \\ b(\tau_0) &\triangleq E_{f_{\mathbf{r}_a}} \left\{ \left[\frac{\partial \ln f_{\mathbf{r}_i}(\mathbf{r} | \tilde{\tau})}{\partial \tilde{\tau}} \right]^2 \right\} \Big|_{\tilde{\tau}=\tau_0} \end{aligned} \quad (11)$$

and where the notation $E_{f_{\mathbf{r}_a}}$ means that the statistical expectation has to be computed using the pdf of the actual received signal \mathbf{r}_a .

It can be shown that, applying (10)-(11) to (6) and using (9), we have

$$\begin{aligned} a(\tau_0) &= \frac{1}{\sigma_z^2} \sum_{n=0}^{N-1} \left[(y[n] - x[n; \tau_0]) \ddot{x}[n; \tau_0] - |\dot{x}[n; \tau_0]|^2 \right] \\ b(\tau_0) &= \frac{1}{\sigma_z^2} \sum_{n=0}^{N-1} |\dot{x}[n; \tau_0]|^2 \end{aligned} \quad (12)$$

where $\dot{x}[n; \tau_0]$ and $\ddot{x}[n; \tau_0]$ represent the samples of the first-order and second-order time-derivative of $x(t; \tau_0)$, respectively, so that the MMCRB reads

$$\begin{aligned} \sigma_{\tilde{\tau}}^2 &\geq MMCRB_d(\tau) \\ &= \frac{\sigma_z^2 \sum_{n=0}^{N-1} |\dot{x}[n; \tau_0]|^2}{\left(\sum_{n=0}^{N-1} \left[(y[n] - x[n; \tau_0]) \ddot{x}[n; \tau_0] - |\dot{x}[n; \tau_0]|^2 \right] \right)^2} \end{aligned} \quad (13)$$

In continuous time (letting $T_s \rightarrow 0$) we have

$$\begin{aligned} MMCRB_c(\tau) &= \\ &= \frac{(N_0/2) \int_0^{T_{obs}} |\dot{x}(t - \tau_0)|^2 dt}{\left(\int_0^{T_{obs}} |\dot{x}(t - \tau_0)|^2 dt - \int_0^{T_{obs}} (y(t) - x(t - \tau_0)) \ddot{x}(t - \tau_0) dt \right)^2} \end{aligned} \quad (14)$$

As a sanity check, in the conventional case of no mismatch, $\tau_0 = \tau$ and $y[n] = x[n; \tau]$, so that (13) falls back to the conventional CRB for delay estimation [2]

$$CRB(\tau) = \frac{N_0/2}{\int_0^{T_{obs}} |\dot{x}(t - \tau_0)|^2 dt} \quad (15)$$

We can also re-cast (14) in terms of mismatch-induced degradation Δ_τ wrt the conventional $CRB_c(\tau)$ [2]:

$$MMCRB_c(\tau) = CRB_c(\tau) \cdot \Delta_\tau, \quad \Delta_\tau \geq 1$$

$$\Delta_\tau \triangleq \frac{1}{\left(1 - \frac{\int_0^{T_{obs}} (y(t) - x(t - \tau_0)) \dot{x}(t - \tau_0) dt}{\int_0^{T_{obs}} |\dot{x}(t - \tau_0)|^2 dt}\right)^2} \quad (16)$$

Considering a very large, symmetric observation window and using Parseval relation, we find after a few computations

$$\Delta_\tau = \left(\frac{\int_{-\infty}^{\infty} (2\pi f)^2 |X(f)|^2 df}{\Re \left\{ \int_{-\infty}^{\infty} (2\pi f)^2 Y(f) X^*(f) e^{j2\pi f \tau_0} df \right\}} \right)^2$$

$$= \left(\frac{\int_{-\infty}^{\infty} |\dot{x}(t)|^2 dt}{\Re \left\{ \int_{-\infty}^{\infty} y(t) \dot{x}(t - \tau_0) dt \right\}} \right)^2 \quad (17)$$

With no mismatch, $\Delta_\tau = 1$. In the general case, the degradation depends on the ratio between the second-order moment of the self-spectrum of $x(t)$ and the second-order moment of the cross-spectrum between $y(t)$ and the re-time-phased signal $x(t - \tau_0)$. From (15)-(16), and assuming that the energy of the mismatched signal $y(t)$ is equal to that of the nominal signal $x(t)$, i.e., $E_y = E_x$, then Δ_τ can also be interpreted as the *degradation* in terms of signal-to-noise-ratio to attain the same estimation accuracy when observing the distorted signal instead of the nominal one.

Until now, we have neglected the dependence of the nominal (as well as the mismatched) signal on the ranging code \mathbf{c} . Assume that we observe a generic real-valued *pilot* ranging signal $x(t; \mathbf{c})$ bearing no information data. Strictly speaking, $x(t - \tau; \mathbf{c})$ is a *finite-power parametric random process* with time-unlimited sample functions depending on a ranging code \mathbf{c} whose chips are iid binary ($\in \{\pm 1\}$) random variables. In our estimation problem, the ranging chips are *nuisance parameters* we wish to get rid of. Calling $\bar{x}(t; \mathbf{c})$ the nominal signal $x(t; \mathbf{c})$ time-shifted by the pseudo-true delay τ_0 , i.e., $\bar{x}(t; \mathbf{c}) \triangleq x(t - \tau_0; \mathbf{c})$, we can show that a result similar to what described in [4] for the conventional CRB holds for the MMCRB as well:

$$M^3CRB(\tau) = MCRB(\tau) \cdot \bar{\Delta}_\tau \quad (18)$$

where the degradation $\bar{\Delta}_\tau$ is now:

$$\bar{\Delta}_\tau = \left(\frac{\int_{-\infty}^{\infty} (2\pi f)^2 S_x(f) df}{\int_{-\infty}^{\infty} (2\pi f)^2 \Re \{ S_{y\bar{x}}(f) \} df} \right)^2, \quad (19)$$

In (19), $S_{y\bar{x}}(f)$ is the power spectral density of $\bar{x}(t; \mathbf{c})$, $S_{y\bar{x}}(f)$ is the cross-power spectrum between $y(t; \mathbf{c})$ and $\bar{x}(t; \mathbf{c})$, and the $MCRB(\tau)$ [4], [6] is

$$MCRB(\tau) = \frac{N_0}{T_{obs} 4\pi^2 \int_{-\infty}^{\infty} f^2 S_x(f) df} \quad (20)$$

We call (18)-(19) the Modified Mismatched CRB or M^3CRB .

A. A simple example - nonlinear distortion

Let us make a simple example of the computation of Δ_τ . We assume that we wish to estimate the propagation delay of the baseband bandlimited pulse $x(t) = \text{sinc}(2Bt)$ where B is the signal bandwidth. Assume that for some reasons the actual received pulse is a distorted version of the original pulse, i.e., $y(t) = x^2(t - \tau)$. This is an instance of the simple general case wherein $y(t) = g[x(t - \tau)]$ where $g[\cdot]$ is a memoryless function, as in the case of a distorting High-Power Amplifier (HPA) onboard a satellite. If both $x(t)$ and the nonlinear characteristics $g[\cdot]$ are even-symmetric (as in this example), then it is easily shown that $\tau_0 = \tau$. In our case, assuming for simplicity $\tau = 0$, we get

$$X(f) = \frac{1}{2B} \text{rect}\left(\frac{f}{2B}\right)$$

$$Y(f) = \frac{1}{2B} \left(1 - \frac{|f|}{2B}\right) \text{rect}\left(\frac{f}{4B}\right) \quad (21)$$

so that

$$\int (2\pi f)^2 |X(f)|^2 df = \pi^2 \frac{2B}{3}$$

$$\int (2\pi f)^2 Y(f) X^*(f) df = \pi^2 \frac{5B}{12} \quad (22)$$

and finally

$$\Delta_\tau = \left(\frac{2\pi^2 B/3}{5\pi^2 B/12} \right)^2 = \frac{64}{25} = 2.56 \quad (23)$$

with a sizeable degradation of the best estimation accuracy, in spite of the relatively mild distortion of the pulse¹.

B. Another simple example - multipath propagation

When a radio signal propagates in an urban environment, the received signal is affected by multipath propagation, i.e., more than one copy of the transmitted signal is received because of the multiple propagation paths, each copy characterized by a different delay and amplitude wrt the line-of-sight (LOS), main path. Assuming for simplicity just two such paths,

$$y(t) = x(t) + a \cdot x(t - \Delta t)$$

$$Y(f) = X(f) [1 + a e^{-j2\pi \Delta t f}] \quad (24)$$

where a and Δt are the relative amplitude and delay of the secondary path, respectively, wrt to the main, LOS path.

¹To be 100% fair, we remark that $y(t)$, after the quadratic nonlinear distortion, carries *less* energy than $x(t)$, in particular $E_x = 1/(2B)$ while $E_y = 1/(3B)$. If we assume the same E/N_0 ratio in the two cases, the degradation is smaller than in (23) by a factor $3/2=1.5$

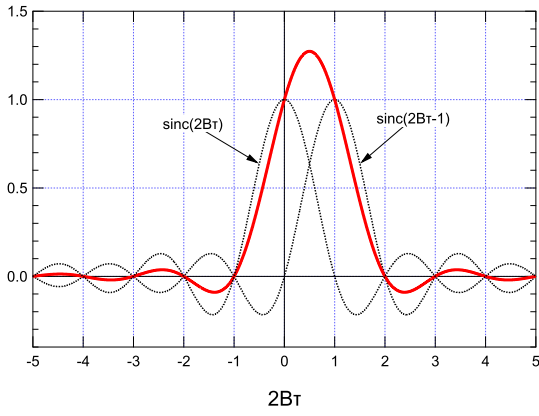


Fig. 1. Derivation of τ_0 with multipath propagation

Assume again, $x(t) = \text{sinc}(2Bt)$ and, as a simple example, $a = 1$ and $\Delta t = 1/(2B)$. Letting for simplicity $\tau = 0$, the pseudo-true value of the delay τ_0 is found as

$$\begin{aligned} \tau_0 &= \underset{\tilde{\tau}}{\text{argmax}} \int y(t)x(t - \tilde{\tau}) dt \\ &= \underset{\tilde{\tau}}{\text{argmax}} \int Y(f)X^*(f) \exp(j2\pi f\tilde{\tau}) df \end{aligned} \quad (25)$$

After a simple computation we find

$$\begin{aligned} &\int Y(f)X^*(f) \exp(j2\pi f\tilde{\tau}) df \\ &= \text{sinc}(2B\tilde{\tau}) + \text{sinc}\left(2B\left(\tilde{\tau} - \frac{1}{2B}\right)\right) \end{aligned} \quad (26)$$

and with the help of Fig. 1 we see that $\tau_0 = 1/(4B)$. The numerator in the expression of Δ_τ is equal to $2B\pi^2/3$ as in the example before, whilst the denominator is now

$$\begin{aligned} &\Re \left\{ \int_{-B}^B (2\pi f)^2 Y(f)X^*(f) e^{j\frac{\pi f}{2B}} df \right\} \\ &= 2 \int_0^B \left(\frac{\pi f}{B}\right)^2 \cdot 2 \cos\left(\frac{\pi f}{2B}\right) df = \frac{8}{\pi} [\pi^2 - 8] B \end{aligned} \quad (27)$$

so that

$$\Delta_\tau = \left(\frac{2\pi^2/3}{\frac{8}{\pi} [\pi^2 - 8]} \right)^2 \simeq 1.91 \quad (28)$$

III. CONCLUSIONS AND FURTHER WORK

The title of this paper contains an open question about the applicability of the M³CRB to satellite positioning. Admittedly, the results presented until now qualify more as a *suggestion* than a final answer. What we have in mind, to be corroborated by further work, is first and foremost the extension of the derivation of the M³CRB to the complex-valued baseband equivalent of a modulated (Radio Frequency) signal, including the introduction of carrier phase-shift on top of the time delay. The reader will in fact have noticed that up to this point all derivations have been carried out on a

real-valued signal – something not 100% realistic in GNSS. This goal will call for the computation of joint as well as marginal M³CRBs for carrier phase and group delay. After this is accomplished, we can derive two fundamental KPIs for the two main problems already tackled in the paper with simple examples, namely:

1. *Computation of the M³CRB on a complete two-ray multipath channel.* In this case, the mismatched model will be a complex-valued extension of (24):

$$y(t) = x(t) + a \cdot x(t - \Delta t)e^{j\theta} \quad (29)$$

where θ is the phase shift of the secondary, reflected path with respect to the LOS path.

2. *Computation of the M³CRB when the RF signal is distorted by an HPA,* whose general description is

$$\begin{aligned} |y(t)| &= \sqrt{P_{OUT,Sat}} g_{AMAM} \left(\left| \frac{x(t)}{\sqrt{P_{IN,Sat}}} \right| \right) \\ \angle y(t) &= \angle x(t) + g_{AMP} \left(\left| \frac{x(t)}{\sqrt{P_{IN,Sat}}} \right| \right) \end{aligned} \quad (30)$$

where $P_{IN,Sat}$, $P_{OUT,Sat}$ are the input/output HPA saturation power, respectively, and $g_{AMAM}(\cdot)$, $g_{AMP}(\cdot)$ are the normalized AM/AM and AM/PM input-output memoryless characteristics, depending on the specific device.

Once the M³CRB on the multipath channel 1) above is computed, the difference $\beta \triangleq \tau_0 - \tau$ will represent a *receiver-independent* KPI having the role of the *multipath error envelope* (MPE) [5] customarily considered in navigation signal design to assess the robustness of a certain signal in comparison with another. In fact, β represents the *best bias* that any estimation algorithm can attain, independent of the particular parameters of a tracking loop or estimation algorithm (discriminator, correlation spacing, coherency, etc.) – it has the same meaning as the MPE, but it is receiver-independent: it is only determined by the intrinsic mismatch.

The same remark applies to optimization of the HPA operating point that can be carried out after 2): the M³CRB on a nonlinear channel will represent a criterion to design the payload for the best ranging accuracy, replacing current KPIs customarily considered in this case, such as the correlation loss.

REFERENCES

- [1] E.D. Kaplan, C.J. Hegarty (Eds.), “*Understanding GPS - Principles and Applications, II Ed.*,” Artech House, Boston, 2006.
- [2] S.M. Kay, “*Fundamentals of Statistical Signal Processing: Estimation Theory*,” Prentice Hall, 1993.
- [3] S. Fortunati, F. Gini, M.S. Greco, C.D. Richmond, “Performance Bounds for Parameter Estimation under Misspecified Models”, *IEEE Signal Processing Magazine*, November 2017
- [4] A. N. D’Andrea, U. Mengali and R. Reggiannini, “The modified Cramer-Rao bound and its application to synchronization problems”, *IEEE Transactions on Communications*, February-April 1994
- [5] P. Teunissen, and O. Montenbruck (eds.), “*Springer Handbook of Global Navigation Satellite Systems*”, Springer, 2017
- [6] M. Luise, F. Zanier, “Multicarrier Signals: A Natural Enabler for Cognitive Positioning Systems”, In: Plass, S., Dammann, A., Kaiser, S., Fazel, K. (eds) *Multi-Carrier Systems & Solutions 2009. Lecture Notes in Electrical Engineering*, vol 41. Springer, Dordrecht.