Robust Sparse Beamforming via Minimax and Maximin SINRs for a Radar Receive Array

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***Abstract*—This paper considers robust sparse beamforming (RSB) designs for a radar receive array via minimax and maximin signal-to-interference-plus-noise ratios (SINRs) criteria (assuming mismatches on the signal-of-interest steering vector and the interference-plus-noise covariance matrices) with a sparsity constraint on the beamvector. We first propose an approximation algorithm for the RSB solution for the minimax SINR problem, where a re-weighted l_1-norm regularization method is exploited to account for the sparsity requirement (by means of a tailored penalty term to the objective) and each problem of the underlying sequence of optimizations is recast into a semidefinite programming (SDP) problem by the strong duality theory. We show how to retrieve an optimal beamvector for the regularized minimax problem from the optimal solution to the SDP. For the RSB solution to the maximin SINR problem, an approximation algorithm is established similarly to the previous situation, but with the difference that the regularized maximin problem is transformed into a second-order cone programming problem. The latter approximation algorithm is computationally lighter than the former when the size of the array is sufficiently large. Simulation examples are presented to demonstrate the improved performance of the proposed two RSB solutions in terms of the normalized beampattern and array output SINR, compared to two existing non-robust sparse beamformers.**

***Index Terms*—Robust sparse beamforming, minimax and maximin SINRs, l_1-norm regularization, semidefinite programming, second-order cone programming.**

I. INTRODUCTION

Robust adaptive beamforming (RAB) techniques have been popular and powerful tools in signal processing applications such as radar, wireless communications, sonar, speech, and so on within the past two or three decades. Among them, the robust sparse beamforming (RSB) approaches are envisioned as viable means to accomplish spatial signal processing due to their capability of achieving large spatial apertures with a reduced hardware complexity (see, e.g., [1] for sparse array beamforming). In a radar signal processing context, the RSB solution is useful to mitigate the effects of harmful interferences while using much less radar channels.

Typically, the RSB is framed as an RAB optimization problem with an additional cardinality constraint on the beamvector, where the RAB optimization problem can be commonly formulated according to minimax or maximin criteria for the array output signal-to-interference-plus-noise ratio (SINR), accounting for mismatches between the presumed and the actual steering vectors (SVs) of the signal-of-interest (SOI), and inaccurate estimation of the interference-plus-noise covariance (INC) matrices [2]. Some RAB problems in terms of minimax and maximin SINRs are nonconvex and hard to solve up to the global optimality [3], and RAB problems with the cardinality constraint are even more challenging.

Nowadays, there are numerous existing non-robust sparse beamforming techniques in open literature [1], [4]–[8], therefore, it is possible to frame RSB optimization problems leveraging on the heritage of the non-robust sparse beamforming techniques. In particular, a re-weighted l_1-norm regularization method has been attractive and widely applied to promote sparsity [4]. For instance, the authors in [5] utilize the re-weighted l_1-norm squared relaxation to search for a sparse transmit beamvector such that the transmit power is minimized subject to each user's quality of service guaranteed in the form of per-user SINR greater than or equal to a threshold in a downlink communication system. The sparsity of the transmit beamvector is ruled through a bisection search over the penalty parameter λ. Besides, a non-robust sparse receive beamforming problem aimed at maximizing the array output SINR for a scenario with multiple SOI sources is studied in [6], and a re-weighted l_1-norm regularization approach is applied by updating the weight vector; Then a large-size semidefinite programming (SDP) relaxation problem for the non-robust sparse beamforming problem with the re-weighted l_1-norm squared term is solved in each iteration [6]. In contrast, in [7], the same non-robust sparse beamforming problem is investigated wherein the re-weighted l_1-norm regularization problem is solved by the alternating direction method of multipliers (ADMM), in order to reduce the computational complexity.

On the other hand, more and more sophisticated approaches have been developed to solve the RAB optimization problem via minimax SINR or maximin SINR criterion in the literature (see, e.g. [2], [3], [9]–[12] and their references). For example, the minimum variance distortionless response (MVDR) robust beamformer can be derived by solving a minimax SINR problem under some convex/nonconvex uncertainty sets for the SOI SV [10]. Moreover, the maximin SINR problem

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seeks a beamvector that maximizes the worst-case SINR over uncertainty sets of the parameters, which includes similarity constraints, double-sided norm constraints, or robust sidelobe level control [10], [11], [13]. These methods highlight the effectiveness of robust optimization in enhancing adaptive beamforming performance under practical uncertainties.

In [14], the authors study the design of an RSB problem that maximizes the worst-case SINR under the uncertainty in the signal's direction of arrival, modeled as a presumed angle together with an error interval. By relaxing the sparsity constraint into a set of linear constraints, the RSB solution is obtained solving a sequence of linear programming problems. In contrast, the RSB solution approaches for optimization problems via minimax and maximin SINR criteria are mentioned in [15].

In this paper, we present the RSB designs by tackling the optimization problems of minimax and maximin SINRs, under convex uncertainty sets for both the SOI SV and the data covariance. To obtain an effective RSB solution for minimax SINR problem, we approximate the original design problem introducing a penalty term proportional to the l_1-norm of a re-weighted beamvector, and reformulate it into an equivalent SDP problem using the strong duality theory [16]. Then, in each iteration of the proposed approximation algorithm, a sequence of small-scale SDP problems are solved for a given positive penalty parameter enforcing the beamvector sparsity. Hence, the RSB solution is obtained by iteratively updating the parameter until the desired sparsity level of the beamvector is achieved. In this context, we also show how to recover an optimal beamvector for minimax SINR problem with a regularized term from the SDP solution via the complementary conditions. As to RSB solution for maximin SINR problem, we employ a similar technique to relax the cardinality constraint and transform the maximin optimization with a regularized term into a second-order cone programming (SOCP) problem. Therefore, the different steps in the algorithm for the RSB solution for maximin SINR problem include solving an SOCP problem to obtain an optimal beamvector, rather than an SDP and finally recovering an optimal beamvector. Simulation results demonstrate that the proposed RSBs achieve higher array output SINRs and improved beampatterns than two existing non-robust sparse beamforming methods.

II. Signal Model And Problem Formulation

Let us consider the SOI from a point signal source impinging on N sensors of a receive narrowband ULA. The receive signal is expressed by

$$\boldsymbol{y}(t) = s(t)\boldsymbol{a} + \boldsymbol{i}(t) + \boldsymbol{n}(t), \tag{1}$$

where the statistically independent components $s(t)\boldsymbol{a}$, $\boldsymbol{i}(t)$, and $\boldsymbol{n}(t)$ are the SOI, interference and noise, respectively, and $s(t)$ and $\boldsymbol{a}$ are the SOI waveform and SV, respectively. The beamformer outputs the signal

$$x(t) = \boldsymbol{w}^H \boldsymbol{y}(t), \tag{2}$$

where $\boldsymbol{w} \in \mathbb{C}^N$ is the beamvector and $(\cdot)^H$ represents the conjugate transpose. Accordingly, the array output SINR can be calculated as

$$\text{SINR} = \frac{\sigma^2 \boldsymbol{w}^H \boldsymbol{a}\boldsymbol{a}^H \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}} \tag{3}$$

where σ^2 is the SOI power and $\boldsymbol{R}_{i+n}$ is the INC matrix.

In practical applications, the INC matrix $\boldsymbol{R}_{i+n}$ is usually unavailable and the SOI SV $\boldsymbol{a}$ can be only inaccurately predefined. Therefore, $\boldsymbol{R}_{i+n}$ and $\boldsymbol{a}$ are respectively replaced in (3) by the data sample covariance matrix

$$\hat{\boldsymbol{R}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{y}(t)\boldsymbol{y}^H(t), \tag{4}$$

and the presumed SOI SV $\hat{\boldsymbol{a}}$ in (3). Then, maximizing the approximate SINR leads to the well-known MVDR beamformer $\boldsymbol{w}^\star = \hat{\boldsymbol{R}}^{-1}\hat{\boldsymbol{a}}$.

However, the MVDR beamformer performance degrades quickly due to the mismatches between the nominal and the actual parameters [9]. In order to address this issue, the RAB techniques employing minimax and maximin SINR criteria [2] appear interesting:

$$\underset{\boldsymbol{a}\in\mathcal{A}_1, \boldsymbol{R}\in\mathcal{B}_1}{\text{minimize}} \quad \underset{\boldsymbol{w}\in\mathcal{W}_1}{\text{maximize}} \quad \frac{|\boldsymbol{w}^H\boldsymbol{a}|^2}{\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}}, \tag{5}$$

and

$$\underset{\boldsymbol{w}\in\mathcal{W}_1}{\text{maximize}} \quad \underset{\boldsymbol{a}\in\mathcal{A}_1, \boldsymbol{R}\in\mathcal{B}_1}{\text{minimize}} \quad \frac{|\boldsymbol{w}^H\boldsymbol{a}|^2}{\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}}, \tag{6}$$

where $\mathcal{A}_1$ and $\mathcal{B}_1$ are the uncertainty sets for the SOI SV $\boldsymbol{a}$ and the INC matrix $\boldsymbol{R}$ (the subscript is dropped here and afterward for notational simplicity), respectively, and $\mathcal{W}_1$ is the set of feasible beamvectors.

In many radar applications (e.g., sparse array radar, target detection and localization, and cognitive radar [3], [6], [8], [14]), a constraint on the beamvector's cardinality could be additionally required in RAB design problems (5) and (6), namely, the feasible set of beamvectors is defined as

$$\mathcal{W}_1 = \{\boldsymbol{w} \in \mathbb{C}^N \setminus \{\boldsymbol{0}\} \mid \|\boldsymbol{w}\|_0 = L\}, \tag{7}$$

where the positive integer L is smaller than N and $\|\boldsymbol{w}\|_0$ means the total number of the nonzero elements of $\boldsymbol{w}$.

III. Robust Sparse Beamforming via Minimax SINR and Maximin SINR Optimizations

In this section, we propose approximation algorithms for RSB solutions for the two design problems.

A. Robust Sparse Beamforming via Minimax SINR Criterion

Suppose that uncertainty set $\mathcal{A}_1$ is modeled as a ball constraint:

$$\mathcal{A}_1 = \{\boldsymbol{a} \mid \|\boldsymbol{a} - \hat{\boldsymbol{a}}\|_2^2 \le \epsilon\}, \tag{8}$$

where $\|\cdot\|_2$ is an l_2-norm for a vector, and $\mathcal{B}$ includes all positive semidefinite matrices satisfying a similarity constraint:

$$\mathcal{B}_1 = \{\boldsymbol{R} \mid \|\boldsymbol{R} - \hat{\boldsymbol{R}}\|_F^2 \le \gamma,\, \boldsymbol{R} \succeq \boldsymbol{0}\}, \tag{9}$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

It is not hard to reexpress problem (5) with the sparsity beamvector constraint as

$$\underset{\boldsymbol{a}\in\mathcal{A}_1,\boldsymbol{R}\in\mathcal{B}_1}{\text{maximize}} \quad \underset{\boldsymbol{w}\in\mathcal{W}_1'}{\text{minimize}} \quad \boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}, \tag{10}$$

where the feasible set $\mathcal{W}_1' = \mathcal{W}_1 \cap \mathcal{W}_0$ with $\mathcal{W}_0 = \{\boldsymbol{w} \mid \Re(\boldsymbol{w}^H\boldsymbol{a}) \geq 1\}$. In order to tackle problem (10), a popular and efficient way is to apply the l_1-norm regularization method and penalize the cost function [4] obtaining the following design problem

$$\underset{\boldsymbol{a}\in\mathcal{A}_1,\boldsymbol{R}\in\mathcal{B}_1}{\text{maximize}} \quad \underset{\boldsymbol{w}\in\mathcal{W}_0}{\text{minimize}} \quad \boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w} + \lambda\|\boldsymbol{b}^{(k)}\odot\boldsymbol{w}\|_1, \tag{11}$$

where $\odot$ denotes the Hadamard product, $\lambda > 0$ is a penalty factor to be updated for the desired sparsity in $\boldsymbol{w}$, and $\boldsymbol{b}^{(k)}$ is the re-weighting vector in the k-th iteration. Specifically, to determine an effective RSB solution to (10), a sequence of problems (11) must be solved, and each of them is handled according to the following steps. First, select a proper λ via a bisection search (see, e.g., [7, Remark 3]). Second, solve maximin problem (11) fixing $(\lambda, \boldsymbol{b}^{(k)})$ for $k \geq 0$ (for any initial point, say $\boldsymbol{b}^{(0)} = \mathbf{1}$), obtaining $\boldsymbol{w}^{(k)}$; hence, update

$$\boldsymbol{b}^{(k+1)} = \mathbf{1} \oslash (|\boldsymbol{w}^{(k)}| + \zeta) \geq \mathbf{0}, \tag{12}$$

and solve (11) with $(\lambda, \boldsymbol{b}^{(k+1)})$ again, until convergence or $k > k_{\max}$. Here, $\mathbf{1}$ is the all-one vector, $\boldsymbol{x} \oslash \boldsymbol{y} = [x_1/y_1, \ldots, x_N/y_N]^T$, $|\boldsymbol{x}| = [|x_1|, \ldots, |x_N|]^T$, and a small $\zeta > 0$ is fixed to guarantee a nonzero denominator. Third, check whether the RSB solution $\boldsymbol{w}^\star$ satisfies the cardinality condition (7), and if it is fulfilled, output this RSB solution for (10); otherwise, repeat the first and second steps.

In the described method, a nontrivial step includes how to solve maximin problem (11), given $(\lambda, \boldsymbol{b}^{(k)})$. In what follows, we show how it can be converted into an SDP and retrieving an optimal solution $\boldsymbol{w}^\star$ from the complementary slackness conditions. To begin with, the inner minimization problem of (11) can be expressed as an SOCP

$$\begin{aligned} \underset{s,\boldsymbol{w},\boldsymbol{t}}{\text{minimize}} \quad & s^2 + \lambda{\boldsymbol{b}^{(k)}}^T\boldsymbol{t} \\ \text{subject to} \quad & \Re(\boldsymbol{a}^H\boldsymbol{w}) \geq 1, \\ & \boldsymbol{t} \geq |\boldsymbol{w}|, \\ & s \geq \|\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{w}\|_2, \end{aligned} \tag{13}$$

where s and $\boldsymbol{t}$ are auxiliary variables. It is not hard to derive the dual problem of (13), another SOCP, as follows

$$\begin{aligned} \underset{r,z,\boldsymbol{y}}{\text{maximize}} \quad & -\frac{r^2}{4} + z & \text{(14a)} \\ \text{subject to} \quad & \lambda\boldsymbol{b}^{(k)} \geq |z\boldsymbol{a} + \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{y}|, & \text{(14b)} \\ & r \geq \|\boldsymbol{y}\|_2, & \text{(14c)} \\ & z \geq 0, & \text{(14d)} \end{aligned}$$

and the corresponding complementary conditions are

$$\begin{aligned} \Re((z\boldsymbol{a})^H\boldsymbol{w}) - z = 0, \quad & \text{(15a)} \\ \boldsymbol{t}^T(\lambda\boldsymbol{b}^{(k)}) + \Re((z\boldsymbol{a} + \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{y})^H\boldsymbol{w}) = 0, \quad & \text{(15b)} \\ 2s - r = 0, \quad & \text{(15c)} \\ rs + \Re(\boldsymbol{y}^H\boldsymbol{R}^{\frac{1}{2}}\boldsymbol{w}) = 0. \quad & \text{(15d)} \end{aligned}$$

Consequently, the maximin problem (11) can be equivalently transformed into the following maximization problem:

$$\begin{aligned} \underset{r,z,\boldsymbol{y},\boldsymbol{a},\boldsymbol{R}}{\text{maximize}} \quad & -\frac{r^2}{4} + z & \text{(16a)} \\ \text{subject to} \quad & \|\boldsymbol{a} - \hat{\boldsymbol{a}}\|_2 \leq \sqrt{\epsilon}, & \text{(16b)} \\ & \|\boldsymbol{R} - \hat{\boldsymbol{R}}\|_F \leq \sqrt{\gamma}, & \text{(16c)} \\ & \boldsymbol{R} \succeq 0, & \text{(16d)} \\ & z \geq 0, & \text{(16e)} \\ & \text{(14b), (14c) satisfied.} & \text{(16f)} \end{aligned}$$

It appears that problem (16) is a nonconvex optimization problem since (14b) now is nonconvex. However, by letting

$$\boldsymbol{u} := z\boldsymbol{a},\ \boldsymbol{v} := \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{y},\ d := r^2, \tag{17}$$

and utilizing the Schur complement lemma, problem (16) can be reformulated as the following SDP:

$$\begin{aligned} \underset{d,z,\boldsymbol{u},\boldsymbol{v},\boldsymbol{R}}{\text{maximize}} \quad & -\frac{d}{4} + z & \text{(18a)} \\ \text{subject to} \quad & \lambda\boldsymbol{b}^{(k)} \geq |\boldsymbol{u} + \boldsymbol{v}|, & \text{(18b)} \\ & \|\boldsymbol{u} - z\hat{\boldsymbol{a}}\|_2 \leq \sqrt{\epsilon}z, & \text{(18c)} \\ & \|\boldsymbol{R} - \hat{\boldsymbol{R}}\|_F \leq \sqrt{\gamma}, & \text{(18d)} \\ & \begin{bmatrix} d & \boldsymbol{v}^H \\ \boldsymbol{v} & \boldsymbol{R} \end{bmatrix} \succeq 0, & \text{(18e)} \\ & z \geq 0. & \text{(18f)} \end{aligned}$$

In other words, the maximin problem (11) is tantamount to SDP problem (18). Suppose that the SDP is solvable and $(d^\star, z^\star, \boldsymbol{u}^\star, \boldsymbol{v}^\star, \boldsymbol{R}^\star)$ is an optimal solution. Then, we have to recover an optimal solution $(\boldsymbol{w}^\star, \boldsymbol{a}^\star, \boldsymbol{R}^\star)$ for the maximin problem (11) leveraging on the optimal solution for the SDP. It is trivial that $(\boldsymbol{a}^\star, \boldsymbol{R}^\star) = (\boldsymbol{u}^\star/z^\star, \boldsymbol{R}^\star)$. For the optimal $\boldsymbol{w}^\star$, we claim the following lemma based on complementary conditions (15).

Lemma III.1 *Suppose that* $(d^\star, z^\star, \boldsymbol{u}^\star, \boldsymbol{v}^\star, \boldsymbol{R}^\star)$ *is an optimal solution for SDP problem* (18) *for given* $(\lambda, \boldsymbol{b}^{(k)})$. *Then, it holds that*

$$\boldsymbol{w}^\star = -\frac{1}{2}(\boldsymbol{R}^\star)^{-1}\boldsymbol{v}^\star \tag{19}$$

is optimal for problem (11) *together with* $(\boldsymbol{a}^\star, \boldsymbol{R}^\star) = (\boldsymbol{u}^\star/z^\star, \boldsymbol{R}^\star)$.

Starting from the aforementioned results, we can now establish the approximation procedure for the design of an effective RSB beamformer $\boldsymbol{w}^\star$ (with $\|\boldsymbol{w}^\star\|_0 = L$) for the

Algorithm 1 Approximation Algorithm for RSB Problem (10)

Input: $\hat{\boldsymbol{R}}$, $\hat{\boldsymbol{a}}$, L, ϵ, γ, λ_L, λ_U, ζ, $k_{\max}$;
Output: An RSB solution $\boldsymbol{w}^\star$ for problem (10);
1: set $\boldsymbol{w} = \boldsymbol{1}$;
2: **while** $\|\boldsymbol{w}\|_0 \neq L$ **do**
3: let $k = 0$ and $\boldsymbol{b}^{(k)} = \boldsymbol{1}$, and set $\lambda = (\lambda_L + \lambda_U)/2$;
4: **while** no convergence and $k \leq k_{\max}$ **do**
5: solve SDP (18), obtaining $(\boldsymbol{v}^\star, \boldsymbol{R}^\star)$;
6: construct $\hat{\boldsymbol{w}} = -(\boldsymbol{R}^\star)^{-1}\boldsymbol{v}^\star/2$ and set $\boldsymbol{w}^{(k)} = \hat{\boldsymbol{w}}$;
7: update weight vector $\boldsymbol{b}^{(k+1)}$ by (12);
8: $k := k + 1$;
9: **end while**
10: $\boldsymbol{w} := \boldsymbol{w}^{(k-1)}$;
11: $\lambda_L := \lambda$, if $\|\boldsymbol{w}\|_0 > L$, or $\lambda_U := \lambda$, if $\|\boldsymbol{w}\|_0 < L$;
12: **end while**
13: output $\boldsymbol{w}^\star = \boldsymbol{w}$.

minimax problem (10) with a sparsity constraint as described in Algorithm 1.

Remark that in step 4, the convergent condition $|f_{(k)} - f_{(k-1)}| \leq 10^{-3}$ is adopted with $f_{(k)} = (\boldsymbol{w}^{(k)})^H \boldsymbol{R}\boldsymbol{w}^{(k)} + \lambda\|\boldsymbol{b}^{(k)} \odot \boldsymbol{w}^{(k)}\|_1$ and $f_{(-1)} = 0$ (suppose no convergence for $k = 0$, i.e., $f_{(0)} > 10^{-3}$), $k_{\max} = 6$ (the same number reported in [6]).

B. Robust Sparse Beamforming via Maximin SINR

For RSB via maximin SINR, problem (6) can be recast as

$$\underset{\boldsymbol{w}\in\mathcal{W}_0}{\text{maximize}} \quad \frac{\underset{\boldsymbol{a}\in\mathcal{A}_1}{\text{minimize}}\ |\boldsymbol{w}^H\boldsymbol{a}|^2}{\underset{\boldsymbol{R}\in\mathcal{B}_1}{\text{maximize}}\ \boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}}, \tag{20}$$

which amounts to (cf. [12], [17])

$$\underset{\boldsymbol{w}\in\mathcal{W}_0}{\text{minimize}} \quad \frac{\boldsymbol{w}^H(\hat{\boldsymbol{R}}+\sqrt{\gamma}\boldsymbol{I})\boldsymbol{w}}{\max^2\{|\boldsymbol{w}^H\boldsymbol{a}|-\sqrt{\epsilon}\|\boldsymbol{w}\|,0\}}. \tag{21}$$

Therefore, problem (21) can be approximated by an SOCP problem with a regularization term

$$\begin{aligned} &\text{minimize} \quad \boldsymbol{w}^H(\hat{\boldsymbol{R}} + \sqrt{\gamma}\boldsymbol{I})\boldsymbol{w} + \lambda\|\boldsymbol{b}^{(k)} \odot \boldsymbol{w}\|_1 \\ &\text{subject to} \quad \Re(\boldsymbol{w}^H\hat{\boldsymbol{a}}) \geq 1 + \sqrt{\epsilon}\|\boldsymbol{w}\|. \end{aligned} \tag{22}$$

Finally, the proposed algorithm for an RSB solution via the maximin problem (6) can be formulated in a way similar to Algorithm 1, but for replacing steps 5 and 6 in Algorithm 1 with one step: solving SOCP problem (22), obtaining a solution $\boldsymbol{w}^\star$ and letting $\boldsymbol{w}^{(k)} = \boldsymbol{w}^\star$.

IV. Simulation Results

In this section, we conduct numerical experiments to validate the effectiveness of the proposed RSB solutions for minimax and maximin SINR problems. The metrics we adopt are the array beampattern and the output SINR for the proposed RSB solutions and two existing non-robust sparse beamformers.

Let us consider a scenario involving a ULA with $N = 12$ omnidirectional sensors evenly spaced at half-wavelength intervals. The noise power at each sensor is set to 0 dB. We assume that two interfering signals, each with an interference-to-noise ratio (INR) of 30 dB, impinge on the sensor array from directions $\pm 30^\circ$. The presumed direction of the SOI is 0°, while its actual arrival direction is 1°. The training sample size T is fixed to 100, and the output results in the figures are averaged over 200 independent simulation runs. In our numerical experiments, we define the sparsity of the beamvectors by setting $L = 4$ out of the N sensors in the ULA.

In addition to the mismatch in the look direction of the SOI SV, we also include SOI SV's discrepancies caused by wavefront distortion in an inhomogeneous medium, which is the same as the setting in [18, Simulation Example 2]. Precisely, independently random increment phase distortions are accumulated by the components of the actual SV, and within every simulation run, the phase increments following a Gaussian distribution $\mathcal{N}(0, 0.03)$ remain unchanged, but with the zero phase distortion in the first element of the SV.

The two existing non-robust sparse beamformers are adopted for the purpose of comparison, namely, the non-robust sparse beamformers proposed in [6] and [7]. In figures, they are denoted by "Non-robust sparse beamformer via SDR" and "Non-robust sparse beamformer via ADMM", respectively. Moreover, the proposed RSB solution for the minimax SINR problem (by Algorithm (1)) and the RSB solution for the maximin SINR problem (stated in subsection III-B), are termed as "RSB via minimax SINR" and "RSB via maximin SINR", respectively. All the four robust beamformers share the same sample data covariance matrix $\hat{\boldsymbol{R}}$ and the radius square $\gamma = 0.1\|\hat{\boldsymbol{R}}\|_F$. The square of spherical radius, ϵ, of the error set for the SOI SVs is set to $0.1N$.

1) Example 1: In this example, the feasible interval for the parameter λ is set as $[\lambda_L, \lambda_U] = [1, 500]$, and $\zeta = 0.01$. Fig. 1 illustrates the normalized beampatterns of our two proposed robust sparse beamformers along with the non-robust sparse beamformers for the same signal-to-noise ratio (SNR) of 5 dB. It can be observed that the two proposed RSB vectors lead to almost the same beampattern and provide lower sidelobe than the other two non-robust approaches while along the interference directions, all the considered methods provide nulls deep enough. Furthermore, the mainlobe peaks of the normalized beampatterns of the RSBs are aligned with the actual arrival direction of the SOI while the non-robust sparse beamformers' mainlobe peaks do not match the SOI's arrival direction clearly. This implies that the robust designs provide some degree of their RSB solutions immunity with respect to mismatchs between the actual and the presumed SOI SVs and between the INC and the data sample covariance matrices.

2) Example 2: In this example, we set $[\lambda_L, \lambda_U] = [0.1, 1000]$ and remain $\zeta = 0.01$. We test how the array output SINR is affected by the SNR in the range $[-10, 20]$ dB. Fig. 2 shows again that the robust designs for the sparse beamformers lead to an improved resilience in terms of the beamformer output SINRs, compared to the existing non-robust designs of sparse beamformers. In addition, an inspection of the figure highlights that the performance of the two proposed RSBs coincides, as per Example 1. It is worth noting that for a fixed sparsity pattern of the beamformers, the equivalence between minimax and maximin SINRs is ensured [2]; particularly, this equivalence holds true at a per-iteration level in the proposed RSB algorithms. And, that is why the same sparsity patterns occurred along the iterations, as seen in our experiments.

We report that the best λ (corresponding to the desired sparsity of the beamvector) can be found in around five iterations as observed in both case studies 1 and 2.

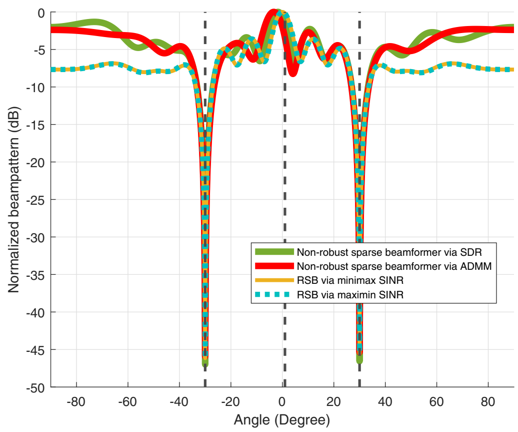

Fig. 1. Normalized beampatterns of the four beamformers.

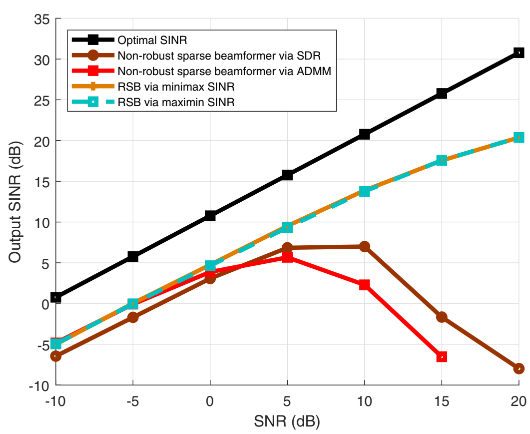

Fig. 2. Output SINR versus input SNR of the four beamformers.

V. Conclusion

We have considered RSB designs via the minimax and maximin SINR criteria with a sparsity constraint on the beamvector under the assumption of a convex uncertainty set of SOI SVs and INC matrix. We have established an approximation algorithm for the minimax SINR problem via a re-weighted l_1-norm regularization method and rewriting the minimax problem with a regularized term into an SDP problem. We have shown how to retrieve an optimal solution for the regularized minimax problem from an optimal solution of the SDP, which is necessary in each iteration of the proposed algorithm. For the maximin SINR problem with a sparsity constraint, the approximation algorithm has been similarly built, but in this case the regularized problem can be rewritten into an SOCP, and thus the computational burden for the RSB solution for maximin SINR problem is lighter than that for minimax SINR counterpart, especially when the size N of the array is sufficiently large. The simulation examples have demonstrated that the RSB solutions via minimax and maximin SINRs outperform the existing two non-robust sparse beamformers in terms of the normalized beampattern and array output SINR.

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