Revisiting Deep Augmented MUSIC Algorithm

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***Abstract*—In this paper, we revisit the hybrid DA-MUSIC model, which combines the classical MUSIC algorithm with deep neural networks to improve the estimation of direction of arrival (DoA). By incorporating Hermitian and positivity constraints into the model covariance matrix, we achieve a performance enhancement. Additionally, we introduce a novel data-driven model that employs recurrent neural networks and multilayer perceptrons, demonstrating superior performance compared to DA-MUSIC across various scenarios.**

***Index Terms*—DoA estimation, data-driven, recurrent neural networks**

I. Introduction

Direction-of-Arrival (DoA) estimation has been an active research area in sensor array processing for several decades. Three distinct approaches are employed to tackle DoA estimation problems: Model-Based (MB), Data-Driven (DD), and Hybrid.

The classical MB approaches involve relationship among quantities, such as the geometry of the sensor array, or statistical assumptions about the model. Among them, MUSIC [1] is a popular algorithm being applicable to many types of arrays. For uniform linear arrays, the Root-MUSIC [2] or ESPRIT [3] algorithms are often preferred since they avoid the 1D peaks finding procedure in the MUSIC pseudo spectrum. For a large number of sensors and snapshots, the performance of MUSIC approaches the Cramer-Rao bound for uncorrelated signals [4]. However, subspace methods show limitations such as a lack of reliability for correlated source signals.

The Data-Driven (DD) approach leverages artificial neural networks for DoA estimation. Recent studies have used fully connected neural networks [5], convolutional neural networks (CNNs) [6], [7] for this task. While these models achieve strong performance, they require large training datasets, high computational resources, and often lack interpretability.

The hybrid models are gray-box models that integrate elements from both the MB and DD approaches. As an example, [8] uses multiple CNNs to learn the MUSIC pseudospectrum using the empirical covariance matrix. Using the pseudospectrum as a label means that this approach continues to face similar challenges as MUSIC in certain scenarios, such as with coherent sources. DA-MUSIC [9], [10] is a state of the art hybrid model that achieves this goal by combining the recurrent neural network (RNN) with the classical MUSIC algorithm. It has demonstrated its superiority over the MB and DD approaches in various challenging conditions, including situations with an unknown number of sources, coherent sources, or broadband signals.

The original DA-MUSIC model estimates a surrogate covariance matrix without enforcing structural constraints. Our contributions in this paper are:

1) Incorporating Hermitian and positivity constraints on the estimated covariance matrix, improving performance.
2) Introducing a novel deep learning model that outperforms DA-MUSIC while maintaining computational efficiency.

Our proposed model has been proven to surpass the DA-MUSIC in the presence of perturbed sensor positions, coherent sources and unknown mutual coupling.

II. Problem Formulation

We consider a scenario where D narrowband plan waves are received on M sensors. Let $\boldsymbol{s}(t) \in \mathbb{C}^D$ denote the vector of transmitted signals and $\boldsymbol{x}(t) \in \mathbb{C}^M$ as the vector of observed signals. They are connected by the following equation:

$$\boldsymbol{x}(t) = \boldsymbol{C}\boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t), \tag{1}$$

where $\boldsymbol{n}(t) \in \mathbb{C}^M$ is an additive white Gaussian noise, $\boldsymbol{C}$ the mutual coupling matrix, $\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1)\ \boldsymbol{a}(\theta_2)\ \dots\ \boldsymbol{a}(\theta_D)]$ the matrix formed by D steering vectors $\boldsymbol{a}(\theta_i)$ and θ_i is the DoA of the i^{th} source for $i = 1 : D$.

Letting (x_i, y_i) for $i = 1 : M$ denote the position of i^{th} sensor and λ the signal wavelength. We have:

$$\mathbf{a}(\theta) = \begin{bmatrix} \exp\left(-j\frac{2\pi}{\lambda}(x_1 \sin\theta + y_1 \cos\theta)\right) \\ \exp\left(-j\frac{2\pi}{\lambda}(x_2 \sin\theta + y_2 \cos\theta)\right) \\ \vdots \\ \exp\left(-j\frac{2\pi}{\lambda}(x_M \sin\theta + y_M \cos\theta)\right) \end{bmatrix}. \tag{2}$$

For instance a uniform linear array (ULA) with inter-element spacing d, we set $x_i = (i-1)d$ and $y_i = 0$ for $i = 1 : M$. Alternatively, for a uniform circular array (UCA) where array elements are distributed uniformly on a circle of radius r, we set $x_i = r\cos\frac{2\pi(i-1)}{M}$ and $y_i = r\sin\frac{2\pi(i-1)}{M}$ for $i = 1 : M$.

The goal of the DoA estimation problem is to determine the directions $\hat{\boldsymbol{\theta}}$ through the observation signals $\boldsymbol{x}(1), \boldsymbol{x}(2), ..., \boldsymbol{x}(T) \in \mathbb{C}^M$.

III. MUSIC Algorithm

The MUSIC algorithm leverages the eigendecomposition of the empirical covariance matrix $\hat{\boldsymbol{R}}_{\boldsymbol{x}}$ of observed signals:

$$\hat{\boldsymbol{R}}_{\boldsymbol{x}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}(t)\boldsymbol{x}(t)^H = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^H. \tag{3}$$

Where $\boldsymbol{D}$ is a diagonal matrix with its diagonal entries arranged in descending order. The last $M-D$ columns of $\boldsymbol{U}$, form a matrix $\boldsymbol{E_N}$ that represents the noise subspace. The columns of $\boldsymbol{E_N}$ are orthogonal to the space spanned by the columns of $\boldsymbol{A}(\boldsymbol{\theta})$. Subsequently, the pseudospectrum $P(\theta)$ is computed across the discretized grid given by:

$$P(\theta) = \frac{1}{\|\boldsymbol{E}_{\boldsymbol{N}}^H\boldsymbol{a}(\theta)\|^2}. \tag{4}$$

The algorithm then identifies the D highest peaks of this pseudo-spectrum $P(\theta)$ computed on a discretized grid. Estimated DoAs are inferred by associating angles with these identified peaks.

IV. DA-MUSIC

DA-MUSIC algorithm is a hybrid approach that combines the classical subspace methods and with deep learning techniques to improve DoA estimation. Unlike conventional MUSIC, DA-MUSIC replaces the empirical covariance estimation step with data-driven model based on a kind of RNN named GRU [11]. The rationale for this approach is that empirical covariance estimation may fail to capture sufficient information when the number of snapshots is limited or when the sources are coherent. Then, the steps of eigendecomposition, selection the eigenvectors corresponding to the $M-D$ smallest eigenvalue, and calculation the pseudospectrum from the selected eigenvectors are left unchanged. Finally, the conventional peak location identification is replaced by a multi-layer perceptron to enable the backpropagation. The computational flow of DA-MUSIC is described in Figure 1.

Inspired by [12], the authors of DA-MUSIC suggest using the Root Mean Square Periodic Error (RMSPE) loss function instead of the traditional RMSE. The RMSPE is defined as:

$$\text{RMSPE}(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}) = \min_{\boldsymbol{P}\in\mathcal{P}_D}\left(\frac{1}{D}\|\text{mod}_\pi(\boldsymbol{\theta} - \boldsymbol{P}\hat{\boldsymbol{\theta}})\|^2\right)^{\frac{1}{2}}, \tag{5}$$

Here, $\mathcal{P}_D$ denotes the set of all permutation matrices of size $D \times D$.

V. Direction of Arrival in the presence of Mutual Coupling

The MUSIC algorithm assumes that each antenna element operates independently without interference from neighboring elements. However, in real-world scenarios, mutual coupling (MC) occurs, leading to undesired interactions between sensors, which distorts the received signals and degrades DoA estimation performance. This section recalls the modified MUSIC algorithm as proposed in [13], [14], [15], for DoA estimation with ULA/UCA under mutual coupling.

If MC matrix $\boldsymbol{C}$ is known, the modified MUSIC spectrum takes the following form:

$$P(\theta) = \frac{1}{\|\boldsymbol{E}_{\boldsymbol{N}}^H\boldsymbol{C}\boldsymbol{a}(\theta)\|^2}. \tag{6}$$

However, in practical applications, $\boldsymbol{C}$ is often unknown. In such cases, definition (6) is replaced by:

$$P(\theta) = \frac{1}{\min\limits_{\boldsymbol{C}\in\mathcal{C}_\alpha}\|\boldsymbol{E}_{\boldsymbol{N}}^H\boldsymbol{C}\boldsymbol{a}(\theta)\|^2}. \tag{7}$$

Where $\mathcal{C}_\alpha = \{\boldsymbol{C} \in \mathbb{C}^{M\times M} | \|\boldsymbol{C}\|_F = \alpha\}$ for some positive constant α. Note that in (7), $\boldsymbol{C}$ depends on θ. The optimization problem in the denominator can be solved analytically by reformulating it as a simple constrained quadratic optimization problem when $\boldsymbol{C}$ is assumed to be Toeplitz (ULA case) or circulant Toeplitz (UCA case). Letting $K = M$ for the ULA case and $K = \lfloor M/2 + 1 \rfloor$ for the UCA case. Consider the matrix $\boldsymbol{T}(\theta) \in \mathbb{C}^{M\times K}$ defined as follows:

1) If the array is ULA,

$$\begin{aligned}\boldsymbol{T}(\theta)_{ij} = \boldsymbol{a}(\theta)_{i+j-1}\mathbf{1}_{\{i+j\leq M+1\}} \\ + \boldsymbol{a}(\theta)_{i-j+1}\mathbf{1}_{\{i\geq j\geq 2\}},\end{aligned} \tag{8}$$

with $\mathbf{1}_{\{P\}} = 1$ if P is true and $\mathbf{1}_{\{P\}} = 0$ otherwise.

2) If the array is UCA,

$$\begin{aligned}\boldsymbol{T}(\theta)_{ij} &= \boldsymbol{a}(\theta)_{i+j-1}\mathbf{1}_{\{i+j\leq M+1\}} \\ &+ \boldsymbol{a}(\theta)_{i-j+1}\mathbf{1}_{\{i\geq j\geq 2\}} \\ &+ \boldsymbol{a}(\theta)_{M+1+i-j}\mathbf{1}_{\{i<j\leq l\}} \\ &+ \boldsymbol{a}(\theta)_{i+j-M-1}\mathbf{1}_{\{2\leq i\leq l,\ i+j\geq M+2\}},\end{aligned} \tag{9}$$

with $l = \lfloor (M+1)/2 \rfloor$.

Here above, $\boldsymbol{T}(\theta)$ is designed such that there exists a vector $\boldsymbol{c}$ satisfying $\boldsymbol{C}\boldsymbol{a}(\theta) = \boldsymbol{T}(\theta)\boldsymbol{c}$. Without loss of generality, we assume that $\boldsymbol{c}^{\boldsymbol{H}}\boldsymbol{c} = 1$. We further assume that MC effects between two sufficiently far sensors are negligible, implying $c_L = c_{L+1} = \cdots = c_K = 0$ for some index $L \leq M - D$. This is a necessary condition for solving the optimization problem in (7). We then denote $\boldsymbol{D}(\theta)$ the first L columns of $\boldsymbol{T}(\theta)$. Then the denominator of (7) rewrites:

$$\min_{\boldsymbol{c}^{\boldsymbol{H}}\boldsymbol{c}=1}\|\boldsymbol{E}_{\boldsymbol{N}}^H\boldsymbol{T}(\theta)\boldsymbol{c}\|^2 = \lambda_{\text{min}}\left(\boldsymbol{Q}(\theta)\right), \tag{10}$$

where $\boldsymbol{Q}(\theta) = \boldsymbol{D}(\theta)^{\boldsymbol{H}}\boldsymbol{E_N}\boldsymbol{E}_{\boldsymbol{N}}^{\boldsymbol{H}}\boldsymbol{D}(\theta)$ and $\lambda_{\text{min}}(\boldsymbol{Q}(\theta))$ is the minimal eigenvalue of $\boldsymbol{Q}(\theta)$. Finally, the modified MUSIC spectrum is expressed as:

$$P(\theta) = \frac{1}{\lambda_{\text{min}}\left(\boldsymbol{Q}(\theta)\right)}. \tag{11}$$

Note that the modified spectrum is always well-defined due to the assumption that $c_L = c_{L+1} = \cdots = c_K = 0$ for some index $L \leq M - D$, which ensures that $\lambda_{\min}(\boldsymbol{Q}(\theta))$ remains strictly positive. And finally, the peaks finding step remains unchanged.

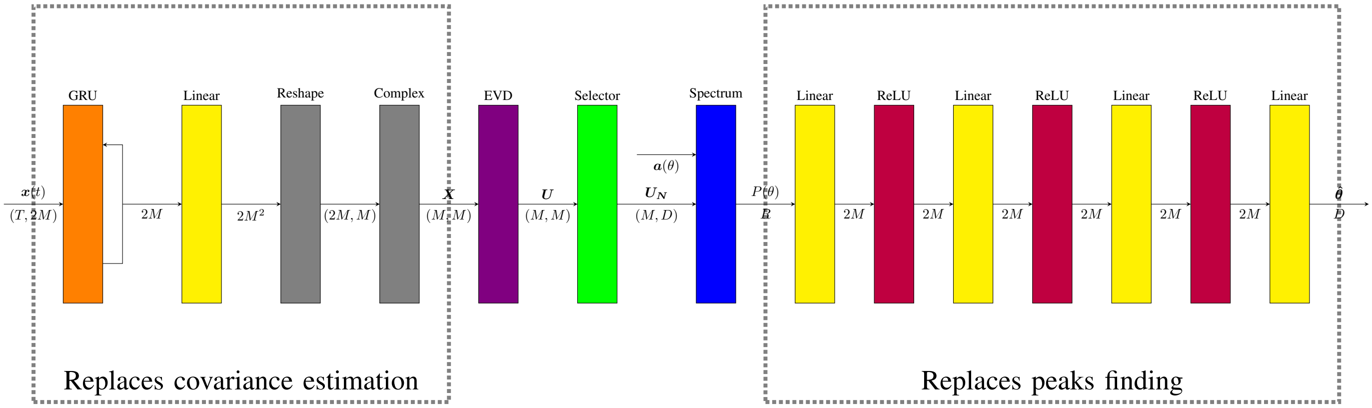

Fig. 1. DA-MUSIC architecture

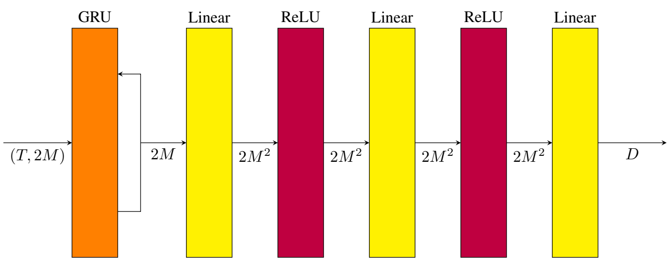

Fig. 2. Proposed DD model

VI. Contribution

A. Modified DA-MUSIC

One issue with DA-MUSIC is that the surrogate covariance estimated from the RNN lacks the inherent properties typical of a covariance matrix, such as Hermitian structure and positivity. We therefore propose to enforce these two special structures to the covariance matrix by adding one transformation $f(\boldsymbol{X}) = \boldsymbol{X}\boldsymbol{X}^{\boldsymbol{H}}$ to the matrix $\tilde{\boldsymbol{X}}$ before taking the eigendecomposition (i.e, just before the fifth block, shown in purple, in the Figure 1)

In the case of ULA, the theoretical covariance matrix not only has Hermitian and positive structure but it also has a Toeplitz structure. A further step would be to enforce both Hermitian and Toeplitz structures on the covariance matrix. But, it appears that this results in slightly inferior performance compared to the original DA-MUSIC method. One possible explanation is that imposing the Toeplitz constraint strongly reduces the number of degrees of freedom of the covariance matrix. In particular, degradations such as mutual coupling or sensor positioning errors also degrade the Toeplitz properties structure that should not be enforced in such situations.

B. Proposed DD model

Although DA-MUSIC has already demonstrated superior performance compared to existing MB and DD models, we aim to further enhance its capabilities as discussed above. We also propose a novel yet simple DD model that combines an RNN and MLP as depicted in Figure 2. In the case where we have few sensors, the proposed model maintains a level of complexity comparable to that of DA-MUSIC while achieving superior performance compared to the aforementioned modified DA-MUSIC approach. Another advantage of this approach is that it eliminates the need for an eigendecomposition step, which relies on an iterative algorithm.

C. Computational Complexity

The computational complexity of the original DA-MUSIC and our proposed modified version is nearly identical for a small number of sensors (e.g., $M = 8, 16$ or 32). However, for larger values of M (e.g., $M = 500$ or 1000), the modified version offers slightly improved computation time. This improvement stems from the fact that the eigenvalue decomposition of a Hermitian matrix is over 10 times faster than that of a general matrix of the same size, as verified using PyTorch [16].

The computational complexity of our proposed DD model is also comparable to that of the original DA-MUSIC. Although the DD model employs $\mathcal{O}(M^4)$ neurons in the final layers, significantly more than the $\mathcal{O}(M^2)$ neurons used in DA-MUSIC, it benefits from full parallelization. Unlike eigendecomposition, which requires a sequential algorithm, our model eliminates this step, enabling efficient parallel execution.

Finally, computing the loss function defined in 5 may appear computationally intractable for large values of D, as a brute-force approach would involve a factorial complexity $\mathcal{O}(D!)$. However, more efficient solutions exist. In particular, the Hungarian algorithm, originally proposed in [17], [18], can solve this type of matching problem with a significantly lower complexity of $\mathcal{O}(D^3)$.

VII. Performance Evaluation

In the first experiment, we evaluated the performance of the algorithms under ideal conditions. As in [10], the experiment used a uniform linear array with $M = 8$ half-wavelength spaced elements. We considered $D = 3$ narrowband, incoherent sources and used $T = 200$ snapshots. The data for learning were generated as in 1. Specifically, for each t, $\boldsymbol{s}(t)$ and $\boldsymbol{n}(t)$ were sampled from complex circular Gaussian distributions

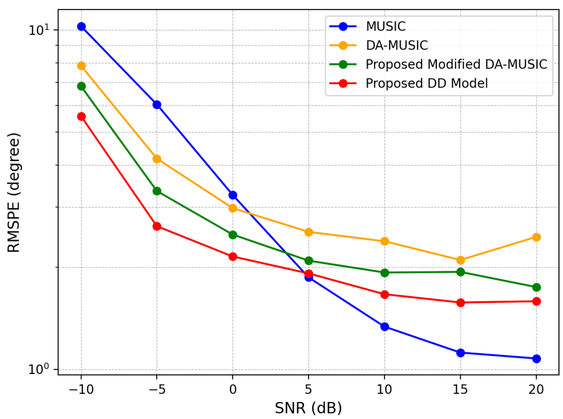

Fig. 3. Ideal conditions, ULA with $M = 8$ sensors, $D = 3$ sources, $T = 200$ snapshots

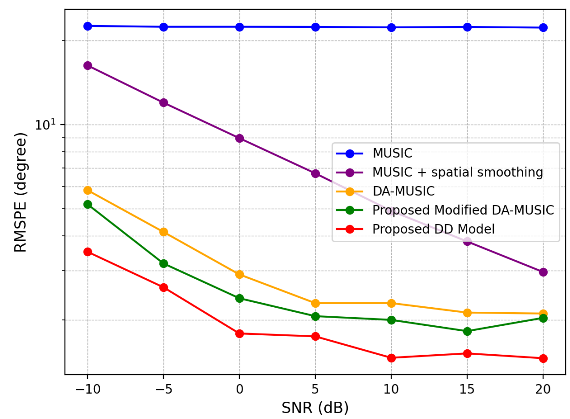

Fig. 4. Coherent sources, ULA with $M = 8$ sensors, $D = 3$ sources, $T = 200$ snapshots

with suitable variances to match the predefined SNR. Then, for each source d, the incident angle θ_d was sampled from the uniform distribution $\mathcal{U}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, ensuring a minimum gap of 0.1 rad between any two angles. This procedure was repeated multiple times to generate a dataset of 20×10^3 samples, which was then split into a training set of 16×10^3 samples and a validation set of 4×10^3 samples. All neural network models were implemented using PyTorch [16] and trained with the Adam optimizer [19] for 200 epochs, with a batch size of 512 and a learning rate of 10^{-3}. Each model was trained multiple times, with a newly generated dataset for each training to select the set of parameters that achieved the best performance. Finally, we evaluated the performance of each model using a test set of 10^5 samples, generated following the same principles as the training set. The result of this experiment is depicted in figure 3. We can easily see that by incorporating the positive Hermitian constraint to the surrogate covariance matrix, DA-MUSIC now performs slightly better. In addition, our proposed DD model, not only beats the original and modified DA-MUSIC, but also the classical MUSIC in the low SNR scenario. Finally, experiments demonstrate that our DD model is four times faster than DA-MUSIC and its modified version and 1.5 times faster than classical MUSIC.

In the second experiment, we evaluated the performance of the models in the presence of coherent sources. Most settings remained the same as in the first experiment, except that in this case we model the coherency of the source signal by taking all entries equal in $\boldsymbol{s}(t)$ for every t. The results of this experiment are illustrated in figure 4. It can be observed as expected that signal coherency degrades the estimation quality of classical MUSIC. In contrast, deep learning methods demonstrate significant advantages, maintaining consistently low RMSPE even in low SNR scenarios and outperforming classical MUSIC in the whole SNR range, even with spatial filtering [20]. We can also see that our modified version of DA-MUSIC brings a small gain compared to the original DA-MUSIC.

In the third experiment, we evaluated the performance of the models under sensor position perturbations. While most settings were identical to those in the first experiment, the key difference was that, in this case, the sensor positions in the ULA were perturbed by uniform noise along the array's direction, and the perturbation values were fixed for the entire dataset and remained unknown to the algorithms. The training and testing procedures were unchanged.The results of this experiment are presented in Figure 5, obtained by averaging the outcomes from 20 different perturbation configurations. We observe that the MUSIC algorithm failed to resolve the DoA even at high SNR levels. In contrast, the data-driven models remained robust in this scenario. Our proposed modified version of DA-MUSIC provided a slight improvement over the original, while our proposed DD model continued to outperform both versions of DA-MUSIC.

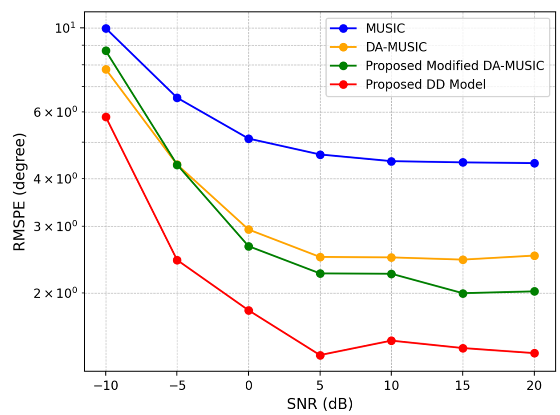

Fig. 5. Perturbed ULA with $M = 8$ sensors, $D = 3$ sources, $T = 200$ snapshots

In the fourth experiment, we evaluated the performance of algorithms under the mutual coupling scenario. Most settings were unchanged from the first experiment, except that we use a uniform circular array of $M = 8$ sensor distributed along a circle of radius $\lambda/2$. We also assumed in this experiment that the mutual coupling matrix as a Toeplitz and circulant

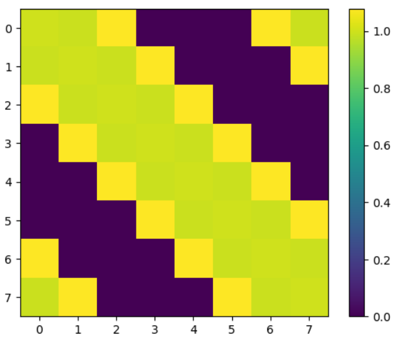

Fig. 6. Structure of mutual coupling matrix

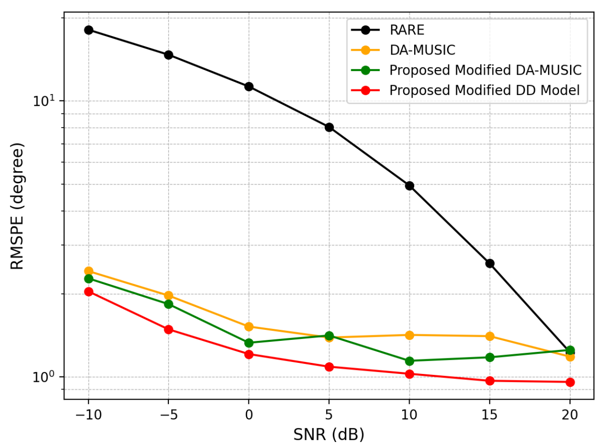

Fig. 7. Presence of mutual coupling, UCA with $M = 8$ sensors, $D = 3$ sources, $T = 200$ snapshots

symmetric structure as depicted in the figure 6. After training and testing using the same procedure as in the first experiment, we obtained figure 7, which compares the performance of the algorithms against the rank-reduction (RARE) algorithm, as discussed in Section V or [13]–[15]. We can see that DA-MUSIC can also be used for DoA estimation under mutual coupling, an aspect does not seem to have been explored in the original paper yet. Moreover, its modified version provides a slight improvement in estimation performance. Finally, our proposed DD model consistently outperforms both versions of DA-MUSIC as well as the RARE algorithm.

VIII. Conclusions

We have introduced an enhanced version of the hybrid DA-MUSIC model for DoA estimation, resulting in improved performance. Additionally, we have developed a purely deep learning model that surpasses not only traditional algorithms like MUSIC, but also a state-of-the-art hybrid DA-MUSIC model in various scenarios, including ideal conditions, sources of coherence, sensor positioning errors and unknown mutual coupling. This novel model maintains acceptable computational complexity compared to DA-MUSIC, particularly when only a few sensors are available.

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