2D GSP-based DOA Estimation for Arbitrary Array

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***Abstract*—Existing direction-of-arrival (DOA) estimation methods based on graph signal processing (GSP) are only applicable for one-dimensional estimation scenario using linear arrays. In this paper, an efficient GSP-based DOA estimation method is proposed applicable for arbitrary array to estimate elevation and azimuth. By constructing cyclic and weighted graphs for given 3D array manifold, the received signal is expressed in graph spectral domain by graph Fourier transform (GFT). Then, by exploiting the graph spectrum, an effective cost function is adopted to construct the graph spatial spectrum, in which a peak search process is performed to obtain the estimated DOA. The simulation results demonstrate that the proposed method achieves computational efficiency under arbitrary array.**

***Index Terms*—2D DOA estimation, arbitrary array, graph signal processing, graph Fourier transform.**

I. Introduction

Direction-of-arrival (DOA) estimation constitutes a fundamental task in numerous sensing and communication systems, including radar systems for autonomous vehicle navigation, underwater sonar arrays for marine exploration, and massive Multipe-Input Multiple-Output (MIMO) antenna configurations in next-generation wireless communications [1]–[5]. Conventional digital signal processing (DSP) based DOA estimation methods usually exploit the regular arrays, such as uniform linear array (ULA) and coprime array (CA). Recent works have advanced sparse array-based DOA estimation through techniques like correlation reconstruction and virtual array interpolation [6]–[8]. However, in practical scenarios, irregular array configurations are often unavoidable, which motivates the need for more flexible approaches that can handle arbitrary array geometries. As an extension of conventional DSP, graph signal processing (GSP) provides an emerging tool to deal with DOA estimation under irregular arrays [9]. By representing sensor arrays with a weighted graph, the array observation can be processed using the graph analysis tools, such as graph Fourier transform (GFT), enabling more flexible manner in array expression and DOA estimation [10]–[12].

In recent years, DOA estimation based on GSP has been studied preliminary. Specifically, [13], [14] used a joint space-time graph to represent the multi-snapshots signal received by a ULA using Kronecker product. By extending traditional beamforming to the graph domain, [15] proposed a single-snapshot DOA estimation method and further investigated the influence of graph structure selection on estimation accuracy. In [16], a GSP-based DOA estimation method for coprime array is explored, achieving superior accuracy to beamforming and MUSIC under low signal-to-noise ratio (SNR) conditions. In our previous work [17], a directional cyclic graph representation of ULA was formulated by defining edge weights as sensors' phase shift, and an efficient cost function was adopted to construct graph spatial spectrum. This method achieved high computational efficiency. However, to the best of our knowledge, existing GSP-based DOA estimation methods are still limited to one-dimensional DOA estimation using a linear array.

In this paper, we propose a two-dimensional DOA estimation framework that integrates graph spectral analysis with graph spatial spectrum construction, which is applicable for arbitrary array configurations. By constructing a directed weighted graph based on the spatial adjacency relationships of array elements, GFT basis are derived from the graph's adjacency matrix and the array observation is transformed into the graph spectral domain. Basing on the analysis of the characteristic of the graph spectrum, an efficient cost function is adopted to construct the graph spatial spectrum. The estimated DOA is obtained by performing spectral peak search in the graph spatial spectrum. Simulation experiments demonstrate that our method achieves high computational efficiency while maintaining comparable resolution performance with conventional method like MUSIC at low SNRs.

II. Signal Model

Consider a 3D arbitrary array with M sensors, which are randomly located in a given range (see an example in Fig. 1). Specifically, the location of the m-th sensor is $\mathbf{p}_m = [p_{mx}, p_{my}, p_{mz}]^{\mathrm{T}}$ in the Cartesian coordinate. Here, $[\cdot]^{\mathrm{T}}$ denotes the transpose. Assume that a narrowband signal impinges on the array from direction (θ, φ), where $\theta \in [0^\circ, 90^\circ]$ and $\varphi \in (-180^\circ, 180^\circ]$ denote the elevation and azimuth of the signal, respectively. In Cartesian coordinate, the DOA (θ, φ) is expressed as

$$\begin{aligned}\mathbf{a} &= [\mathrm{a}_x, \mathrm{a}_y, \mathrm{a}_z]^{\mathrm{T}} &(1)\\ &= [\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta]^{\mathrm{T}}. &(2)\end{aligned}$$

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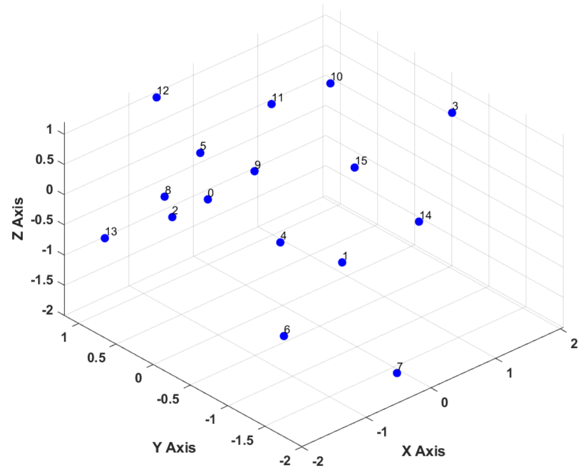


Fig. 1. An example of the 3D arbitrary array. The blue point denotes the sensors, and the number close to the sensor denotes the index of the sensor.

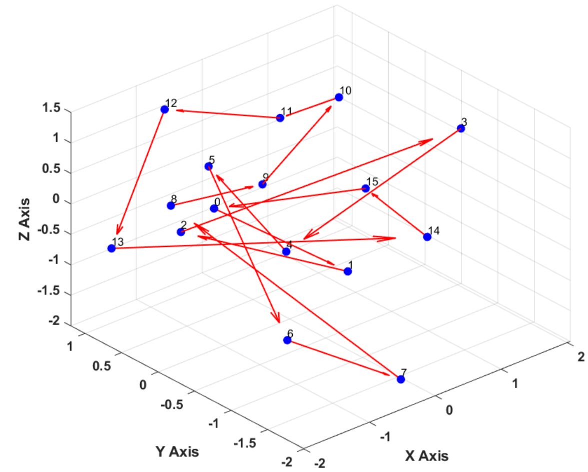


Fig. 2. A cyclic graph representation of an array where the edge connections can be constructed independently of any predefined sequential order.

Thus, the array observation data can be expressed as

$$\mathbf{x}(t) = \boldsymbol{a}s(t) + \mathbf{n}(t), \tag{3}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_M(t)]^{\mathrm{T}}$ contains the observation of M sensors, $\mathbf{n}(t)$ consists of the noise, $s(t)$ is the source waveform, and

$$\boldsymbol{a} = [1, e^{j\frac{2\pi}{\lambda}(\mathbf{p}_2-\mathbf{p}_1)^{\mathrm{T}}\mathbf{a}}, \cdots, e^{j\frac{2\pi}{\lambda}(\mathbf{p}_M-\mathbf{p}_1)^{\mathrm{T}}\mathbf{a}}]^{\mathrm{T}} \tag{4}$$

is the steering vector. Here, j denotes the imaginary unit, and λ is the wavelength.

III. Proposed Algorithm

Different from the conventional array signal expression in DOA estimation, in GSP, each sensor is treated as a directed weighted graph vertex, while the weights on the edges connecting different sensors denote the phase difference (see an example in Fig. 2).

By formulating the graph for the array configuration, the adjacency matrix for a specific direction grid $(\tilde{\theta}, \tilde{\varphi})$ can be expressed as

$$\mathbf{W} = \begin{bmatrix} \mathbf{0}_{M-1} & e^{j\frac{2\pi}{\lambda}(\mathbf{p}_M-\mathbf{p}_1)^{\mathrm{T}}\tilde{\mathbf{a}}} \\ \mathrm{diag}(\boldsymbol{\sigma}) & \mathbf{0}_{M-1}^{\mathrm{T}} \end{bmatrix}, \tag{5}$$

where $\tilde{\mathbf{a}} = [\sin\tilde{\theta}\cos\tilde{\varphi}, \sin\tilde{\theta}\sin\tilde{\varphi}, \cos\tilde{\theta}]$ denotes the Cartesian expression of the direction grid,

$$\boldsymbol{\sigma} = \left[e^{j\frac{2\pi}{\lambda}(\mathbf{p}_2-\mathbf{p}_1)^{\mathrm{T}}\tilde{\mathbf{a}}}, \cdots, e^{j\frac{2\pi}{\lambda}(\mathbf{p}_M-\mathbf{p}_{M-1})^{\mathrm{T}}\tilde{\mathbf{a}}}\right], \tag{6}$$

and $\mathrm{diag}(\boldsymbol{\sigma})$ denotes a diagonal matrix. After obtaining the adjacency matrix, we can adopt GFT to convert the array observation $\mathbf{x}$ into the graph spectral domain as

$$\mathbf{x}_f = \mathbf{V}^{-1}\mathbf{x}, \tag{7}$$

where $\mathbf{V}^{-1}$ is the GFT matrix obtained from eigendecomposition of the adjacency matrix, i.e.,

$$\mathbf{W} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}. \tag{8}$$

Here, $\boldsymbol{\Lambda}$ is a diagonal matrix composed of eigenvalues.

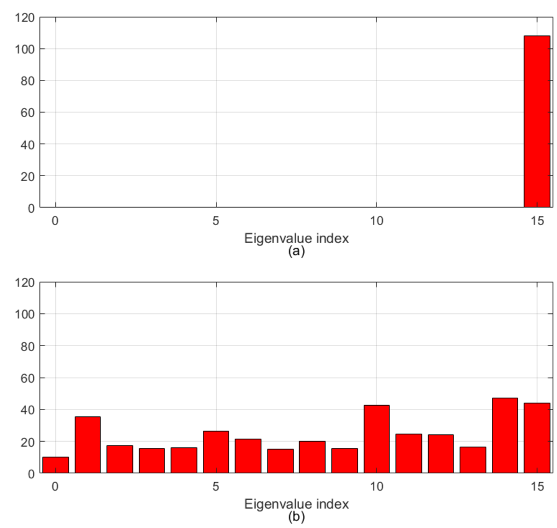


Fig. 3. Graph spectrum of array in Fig. 1 without noise for different cases. (a) Grid $(\tilde{\theta} = 45°, \tilde{\varphi} = -120°)$ matches the true DOA. (b) Grid $(\tilde{\theta} = 60°, \tilde{\varphi} = 30°)$ does not match the true DOA.

The resulting graph spectrum $\mathbf{x}_f$ depends on the relationship between the grid $(\tilde{\theta}, \tilde{\varphi})$ and direction of the target. Specially, consider a target in the direction $(\theta = 45°, \varphi = -120°)$ in space under noise-free conditions. When performing GFT using the matrix $\mathbf{V}^{-1}$ corresponding to the matched grid $(\tilde{\theta} = 45°, \tilde{\varphi} = -120°)$, the resulting graph spectrum shown in Fig. 3(a) exhibits an obvious peak. In contrast, when employing a incorrect grid point $(\tilde{\theta} = 60°, \tilde{\varphi} = 30°)$, the peak becomes submerged in the background components as

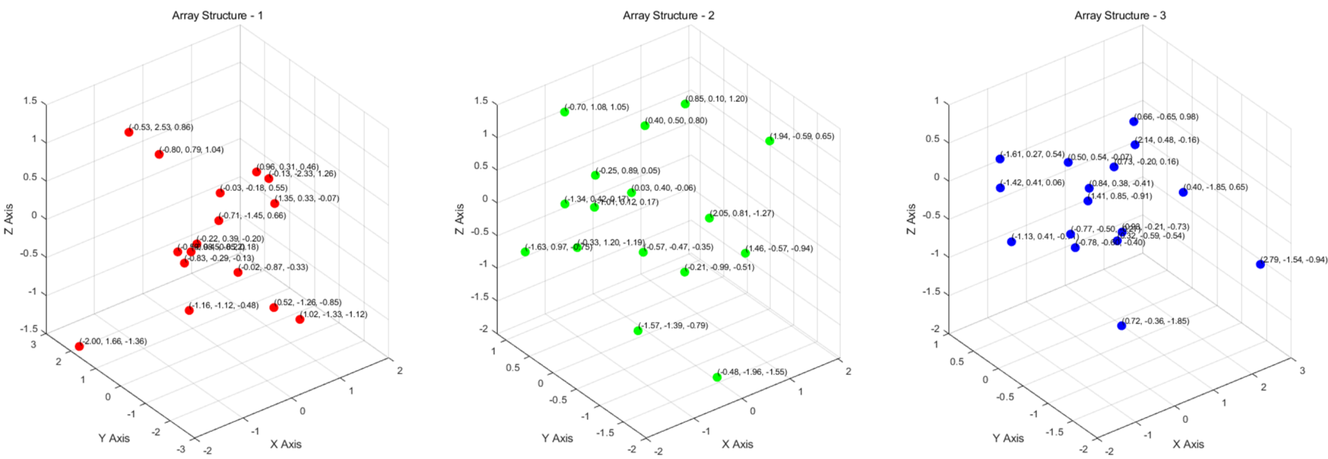


Fig. 4. Three array structures for simulation.

demonstrated in Fig. 3(b). Actually, it can be observed that $\mathbf{W}$ always has the unit eigenvalue. When the grid matches the true DOA, we have

$$\boldsymbol{a}s(t) = \mathbf{W}\boldsymbol{a}s(t), \tag{9}$$

such that the graph frequency spectrum will concentrate on the component corresponding to unit eigenvalue. Otherwise, when the grid does not match the true DOA, the resulting graph spectral does not concentrate on the component of unit eigenvalue.

Therefore, we can define a cost function

$$f(\tilde{\theta}, \tilde{\varphi}) = \frac{1}{N}\sum_{n=1}^{N} \mathbf{v}^{\mathrm{H}}(\tilde{\theta}, \tilde{\varphi})\mathbf{x}(n), \tag{10}$$

where $\mathbf{v}(\tilde{\theta}, \tilde{\varphi})$ is the eigenvector of $\mathbf{W}$ corresponding to the unit eigenvalue, $[\cdot]^{\mathrm{H}}$ denotes the conjugate transpose, and $\mathbf{x}(n)$ is the array observation of the n-th snapshot. Let $\mathbf{D_g} = \{(\tilde{\theta}_1, \tilde{\varphi}_1), (\tilde{\theta}_2, \tilde{\varphi}_2), \cdots, (\tilde{\theta}_{N_g}, \tilde{\varphi}_{N_g})\}$ be the dictionary of the grids of interest, where the number N_g is commonly much larger than the number of sensors M. The graph spatial spectrum, composed of the $f(\tilde{\theta}, \tilde{\varphi})$ of each grid in $\mathbf{D_g}$, serves as the foundation for DOA estimation. The final estimate $(\hat{\theta}, \hat{\varphi})$ is then determined by identifying the dominant peak within this spectrum through a search process, i.e.,

$$(\hat{\theta}, \hat{\varphi}) = \underset{(\tilde{\theta}, \tilde{\varphi}) \in \mathbf{D_g}}{\operatorname{argmax}}\, f(\tilde{\theta}, \tilde{\varphi}). \tag{11}$$

Obviously, once the eigenvectors $\mathbf{v}$ for each grid are obtained, the proposed algorithm only requires vector product computations when constructing the graph spatial spectrum, thereby achieving high computational efficiency.

IV. Simulation Results

In this section, we evaluate the proposed GSP-based DOA estimation algorithm. We randomly generated three array structures, illuminated in Fig. 4, and conduct Monte Carlo trials on these arrays under varying SNRs and snapshot numbers to evaluate the performance of the algorithms. In the following simulations, the number of array elements is $M = 16$, the true DOA is $\theta = 45°$ and $\varphi = -120°$, the dictionary of grids is generated with a step size of $0.5°$ for both the azimuth and the elevation, and the source waveform is generated randomly following the standard normal distribution. The root mean square error (RMSE)

$$\mathrm{RMSE} = \sqrt{\frac{1}{2\mathcal{L}}\sum_{l=1}^{\mathcal{L}}\Big((\hat{\theta}_l - \theta_l)^2 + (\hat{\varphi}_l - \varphi_l)^2\Big)}, \tag{12}$$

and average runtime are selected as the metrics in our simulation. Here, $\mathcal{L} = 1000$ is the number of Monte Carlo trials, θ_l and φ_l are the angles in the l-th Monte Carlo trial, $\hat{\theta}_l$ and $\hat{\varphi}_l$ are their estimates.

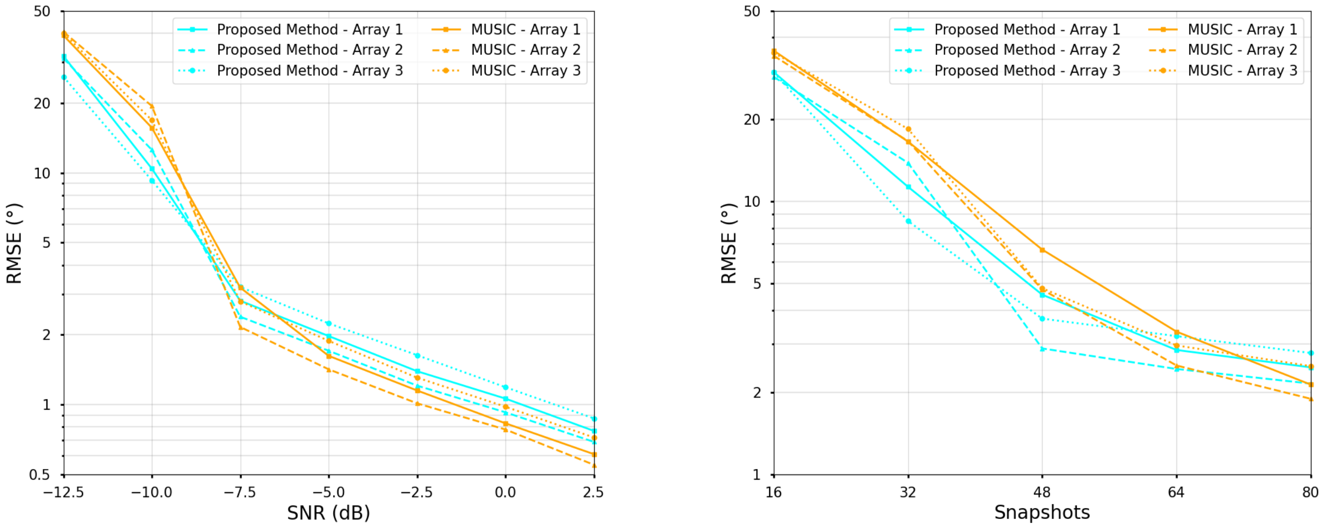
TABLE I
Average Runtime of the Two Algorithms Across Different Snapshot Numbers, Averaged Over 3000 Monte Carlo Tests (1000 Tests per Array Structure)

Number of Snapshots	Proposed Method (ms)	MUSIC (ms)
16	17.80	297.9
32	34.10	298.3

Fig. 5(a) illustrates the RMSE performance versus SNR with $N = 32$ snapshots across three array configurations. It is observed that for all array configurations, as SNR increases from -12.5 dB to 2.5 dB, the RMSE of direction estimates monotonically decreases for all cases. Notably, the proposed algorithm shows better performance than MUSIC algorithm in low SNR situation. As the SNR increases, the proposed algorithm still has the comparable performance to MUSIC.

As shown in Fig. 5(b), the proposed algorithm outperforms MUSIC when the number of snapshots is small, indicating the advantage of the proposed method in the limited snapshots scenarios.

Next, we evaluate the computational complexity of the two algorithms with different snapshots. As shown in Table I, the



(a) (b)

Fig. 5. RMSE of the proposed method and MUSIC. (a) RMSE versus SNR for $N = 32$. (b) RMSE versus snapshots for $\mathrm{SNR} = -10$ dB.

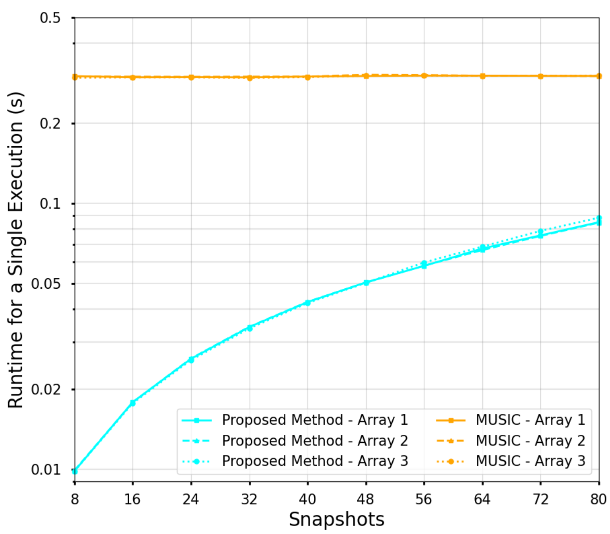


Fig. 6. Average runtime of the two algorithms versus the number of snapshots.

proposed algorithm is about 16 times faster than MUSIC with 16 snapshots, while maintaining accuracy comparable to that of MUSIC. Meanwhile, Fig. 6 illustrates that the runtime of the proposed method increases gradually with the number of snapshots yet remains significantly lower than that of MUSIC, demonstrating its computational efficiency. Furthermore, the transformation basis used in the proposed algorithm depends solely on the array structure and can be precomputed and stored offline, while MUSIC requires the generation of the noise subspace based on the array observation, necessitating online computation.

V. Conclusions

In this paper, we propose a fast GSP-based DOA estimation method applicable for arbitrary array structures in space. By modeling the array as cyclic and weighted graphs, we obtain the GFT basis via eigendecomposition of the adjacency matrices and transform the received signal into the graph spectral domain through GFT. Leveraging characteristics of the graph spectrum, we formulate a cost function to construct the graph spatial spectrum, from which DOA is estimated through peak search. Simulation results demonstrate that the proposed algorithm significantly outperforms MUSIC in computational efficiency while achieving comparable accuracy under low SNR conditions. In future work, we will further investigate the GSP-based DOA estimation methods in more complex scenarios.

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