Low-Rank Approximation for ISAC with Constrained Computational Complexity

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***Abstract*—Joint sensing and communication have become increasingly important research topics for current and future generations of wireless networks. These networks acquire and store a large amount of data that does not provide helpful information about the target location, for instance. As a contribution, we propose a rank-reduction strategy aiming at reducing the computational complexity in the target detection task. The simulation results show that the proposed solutions are very effective when considering the tradeoff between detection and implementation costs.**

***Index Terms*—OTFS, ISAC, Approximated Computation, Randomized Computation**

I. Introduction

It is envisaged that for future networks, such as 6G networks, communication devices should be capable of sensing the environment as a complementary feature [1]–[3].

Integrated sensing and communication (ISAC) has been proposed using different modulation techniques such as the well-known orthogonal frequency division multiplex (OFDM) [4] and most recently orthogonal time frequency space (OTFS) [2] by exploiting the delay-Doppler (DD) domain. In high-mobility communication scenarios, where the Doppler effect hinders communication for OFDM systems, DD domain modulation offers clear advantages in terms of information retrieval, and its key concepts are thoroughly explored in [5].

Energy efficiency is a general problem in engineering and should be considered even for devices without power source limitations. It is known that the reduction in power consumption scales with the number of devices, and a seemingly small reduction has a huge impact on a large scale. Data selection is shown to be effective in reducing the computational burden of the matrix product [6]–[8], enabling many machine learning applications mainly when dealing with Big Data [8].

This work proposes using a matrix product approximation for target detection in ISAC with constrained computational complexity. Many multicarrier systems have a fixed number of subcarriers, resulting in unnecessary range-Doppler resolution for low-budget applications where sensing is a secondary feature of the system. We will see that much of the accumulated data is redundant for information on the targets; hence, we propose methods to select data before performing the cross-correlation at the receiver, achieving an approximate sensing of the environment. The proposed method enables a compromise between target detection probability and computational burden.

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We divided the work into four more sections. In Section II, we introduce the OTFS communication system model used for ISAC. This is followed by Section III, where we explore the integration between the DD domain and the range-Doppler matrix and introduce the propositions for approximated sensing. In Section IV, we present and discuss the results of a numerical simulation for the presented method and compare these results with a method without constrained computation. Finally, Section V concludes the work.

II. System Model

Following the description of the OTFS modulation system [5], [9], [10], let $\mathbf{X}_{\text{DD}} \in \mathbb{C}^{M\times N}$ be the transmitted information frame in the DD domain, represented as $\mathbf{X}_{\text{DD}} = \begin{bmatrix}\mathbf{x}(0) & \mathbf{x}(1) & \cdots & \mathbf{x}(N-1)\end{bmatrix}$, where N is the number of transmitted blocks, M is the number of samples per transmitted block, and $\mathbf{x}(n) \in \mathbb{C}^M$ is the n-th transmitted block of information. As presented in [5], one can use the inverse discrete Zak transform (IDZT) to achieve the OTFS modulation from the DD domain, which results in

$$\mathbf{s} = \text{vec}\left(\mathbf{X}_{\text{DD}}\mathbf{F}_N^{-1}\right), \tag{1}$$

where $\mathbf{s} \in \mathbb{C}^{MN}$ is the discrete-time transmitted signal, $\mathbf{F}_N^{-1} \in \mathbb{C}^{N\times N}$ is the inverse discrete Fourier Transform (IDFT) matrix, and $\text{vec}(\cdot)$ is a vectorization operator which vectorize a matrix in a column-wise stack, i.e., each column is stacked according to the sequence of columns in the matrix.

The continuous-time transmitted signal is reconstructed using an interpolation filter with impulse response $g(t)$, hence

$$s(t) = \sum_{k=0}^{MN-1} \bar{s}(k)g(t-kT_s), \tag{2}$$

where $\bar{s}(k)$ is the k-th element of $\mathbf{s}$, $T_s \in \mathbb{R}$ is the sampling period, and t is the continuous time index for the instant $t \in \mathbb{R}$.

In the radar context, the received signal is composed of echoes and can be modeled similarly to a multipath channel for communication in high-mobility scenarios. Considering a channel with P targets, the p-th echo path is modeled by a set of three parameters: a complex gain, $h_p \in \mathbb{C}$, that represents the random phase and gain of the arriving signal, a delay $\tau_p \in \mathbb{R}$, which models the time between transmission and reception, and a Doppler shift, $\nu_p \in \mathbb{R}$, caused by the relative velocity between the transmitter and the receiver, or a cluster of objects between transmission and reception. We can model the channel in baseband as the following

$$h(\tau, \nu) = \sum_{p=0}^{P-1} h_p \delta(\tau - \tau_p)\delta(\nu - \nu_p), \tag{3}$$

with $\tau_p < T \; \forall \; p$, and $|\nu_p| < \frac{\Delta f}{2} \; \forall \; p$, where $\delta(t)$ is the Dirac delta function, $T = MT_s$ is the block duration, $\Delta f = \frac{1}{T}$ is the spacing between subcarriers, and $|\cdot|$ is the absolute value operator. In the delay-time domain, the channel is presented as

$$\begin{aligned} h(\tau, t) &= \int_{-\infty}^{\infty} h(\tau, \nu) e^{j2\pi\nu(t-\tau)} d\nu \\ &= \sum_{p=0}^{P-1} h_p e^{j2\pi\nu_p(t-\tau)} \delta(\tau - \tau_p). \end{aligned} \tag{4}$$

Therefore, at the receiver, the signal is given by the linear convolution of the transmitted signal with (4), resulting in

$$\begin{aligned} r(t) &= \int_{-\infty}^{\infty} h(\tau, t) s(t - \tau) d\tau + w(t) \\ &= \sum_{p=0}^{P-1} s(t - \tau_p) h_p e^{j2\pi\nu_p(t-\tau_p)} + w(t), \end{aligned} \tag{5}$$

where $w(t) \sim \mathcal{N}_{\mathbb{C}}(0, \sigma_w^2)$ is additive noise. Then, $r(t)$ is converted by an analog-to-digital (AD) converter, and the samples are stacked column-wise into a matrix $\mathbf{Y} \in \mathbb{C}^{M \times N}$ from which a row-wise discrete Fourier transform (DFT) completes the discrete Zak transform (DZT), resulting in the DD domain received signal,

$$\begin{aligned} \mathbf{Y}_{\text{DD}} &= \mathbf{Y}\mathbf{F}_N \\ &= \begin{bmatrix} \mathbf{y}_{\text{DD}}(0) & \mathbf{y}_{\text{DD}}(1) & \cdots & \mathbf{y}_{\text{DD}}(M-1) \end{bmatrix}^T, \end{aligned} \tag{6}$$

where $\mathbf{y}_{\text{DD}}(m) \in \mathbb{C}^N$ is a vector containing the elements of the m-th row of $\mathbf{Y}_{\text{DD}} \in \mathbb{C}^{M \times N}$.

For communication, an equalization process is required to retrieve the transmitted information $\mathbf{X}_{\text{DD}}$ from the matrix $\mathbf{Y}_{\text{DD}}$, this is extensively explored by other authors [5]. However, in sensing applications, we use signal processing techniques to recover range and Doppler information of possible targets [2]–[4].

III. Range-Doppler Sensing in DD Domain

It is known that the range-Doppler matrix, used for radar sensing, is identical to the demodulation process in DD communication [2], [3]. Moreover, one can reduce the necessary post-processing for range and Doppler retrieval to a simple matched filtering process by selecting the correct waveform [2], [3].

From (5), the channel effect in the transmitted signal is perceived as the sum of repetitions of the transmitted signal, according to the number of paths (or targets), shifted in delay by τ_p and in Doppler by ν_p and weighted by a complex gain h_p. Let us begin our analysis by considering an information matrix with a single column activated,

$$\mathbf{X}_{\text{DD}} = \begin{bmatrix} \mathbf{x}(0) & \mathbf{0}_{M \times (N-1)} \end{bmatrix}, \tag{7}$$

where $\mathbf{x}(0) = \begin{bmatrix} x(0) & x(1) & \cdots & x(M-1) \end{bmatrix}^T$ and for the time being, to simplify the analysis, we disregard additive noise and consider quantized values for ν_p and τ_p[1] with ideal pulse shaping and even values for N. Then, the received signal in the DD domain is defined as

$$\mathbf{y}(n) = \begin{cases} \mathbf{0}_{M \times 1}, & n \neq \frac{\Delta_f}{2\nu_p} + \frac{N}{2} \\ \mathbf{x}_p, & n = \frac{\Delta_f}{2\nu_p} + \frac{N}{2} \end{cases}, \quad p = 0, \ldots, P-1, \tag{8}$$

where $\mathbf{y}(n) \in \mathbb{C}^M$ is the n-th column of $\mathbf{Y}_{\text{DD}}$ and $\mathbf{x}_p \in \mathbb{C}^M$ is $\mathbf{x}(0)$ weighted by h_p and cyclically shifted τ_p samples, i.e.,

$$\mathbf{x}_p = h_p \begin{bmatrix} x(M - \tau_p) & \cdots & x(0) & \cdots & x(M - \tau_p - 1) \end{bmatrix}^T. \tag{9}$$

Therefore, in this simplified scenario, the Doppler information is directly extracted from the matrix by exploring which column is not null, while the delay information can be extracted employing the cross-correlation of $\mathbf{x}(0)$ with each column of $\mathbf{Y}_{\text{DD}}$.

The cross-correlation is similar to a filtering process which can be operated using the matrix product [11] of a Toeplitz matrix $\mathbf{X}_{\text{band}} \in \mathbb{C}^{M \times M}$ and the matrix $\mathbf{Y}_{\text{DD}}$. The matrix $\mathbf{X}_{\text{band}}$ is generated using $\mathbf{x}^*(0)$, as follows,

$$\begin{aligned} \mathbf{X}_{\text{band}} &= \begin{bmatrix} x^*(0) & x^*(1) & \cdots & x^*(M-1) \\ x^*(M-1) & x^*(0) & \cdots & x^*(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ x^*(1) & x^*(2) & \cdots & x^*(0) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x}_{\text{band}}(0) & \mathbf{x}_{\text{band}}(1) & \cdots & \mathbf{x}_{\text{band}}(M-1) \end{bmatrix}, \end{aligned} \tag{10}$$

where $\cdot^*$ indicates the complex conjugate of a number and $\mathbf{x}_{\text{band}}(m) \in \mathbb{C}^M$ is a vector corresponding to the m-th column of $\mathbf{X}_{\text{band}}$. The resulting matrix containing the desired estimates is achieved with the product $\mathbf{Y}_{\text{des}} = \mathbf{X}_{\text{band}}\mathbf{Y}_{\text{DD}}$, with $\mathbf{Y}_{\text{des}} \in \mathbb{C}^{M \times N}$.

One can ensure satisfactory cross-correlation results by selecting a proper sequence for $\mathbf{x}(0)$. The Zadoff-Chu (ZC) sequence is an example of a complex code with desirable autocorrelation characteristics [12]; another well-known possibility is the Barker code, as presented in [13].

Conversely, additive noise hinders reception, which affects $\mathbf{y}(n) \; \forall \; n$. The effects of pulse shaping are perceived as a spread of the pulses through the entire lattice of the frame [14], and the non-quantized values of τ_p and ν_p result in imprecise estimates of target delay and Doppler because the peak of

[1]The quantization ensures that $\frac{\Delta_f}{2\nu_p} \in [-\frac{N}{2}, \frac{N}{2} - 1]$ and $\frac{\tau_p}{T_s} \in [0, M-1]$, for $p = 0, \ldots, P-1$.

the pulse main lobe is located in between samples of the DD domain rectangle.

Despite being full-rank, after the simplified analysis, it is clear that the matrix $\mathbf{Y}_{\text{DD}}$ is indeed sparse and can be approximated to a low-rank matrix without major loss of information. Hence, we propose using an approximated matrix product for target estimation with reduced computational burden [6]–[8].

A. Approximated Sensing

As any matrix product, the product $\mathbf{X}_{\text{band}}\mathbf{Y}_{\text{DD}}$ can be interpreted as the sum of rank-one matrices resulting from the outer product of the columns of $\mathbf{X}_{\text{band}}$ with the rows of $\mathbf{Y}_{\text{DD}}$. Let $\mathbf{x}_{\text{band}}(m) \in \mathbb{C}^M$ be the vector containing the elements of the m-th column of $\mathbf{X}_{\text{band}}$ and $\mathbf{y}_{\text{DD}}(m) \in \mathbb{C}^N$ the vector containing the elements of the m-th row of $\mathbf{Y}_{\text{DD}}$, we can write $\mathbf{Y}_{\text{des}}$ as follows

$$\mathbf{Y}_{\text{des}} = \sum_{m=0}^{M-1} \mathbf{x}_{\text{band}}(m)\mathbf{y}_{\text{DD}}^T(m) = \sum_{m=0}^{M-1} \mathbf{Y}_m. \quad (11)$$

Let $||\mathbf{x}||_2 = \left(\sum_{n=0}^{N-1} |x(n)|^2\right)^{\frac{1}{2}}$ be the ℓ_2-norm, defined to any $\mathbf{x} \in \mathbb{C}^N$ vector, and $||\mathbf{x}||_1 = \sum_{n=0}^{N-1} |x(n)|$ its ℓ_1-norm. Following the approach from [6], we want to estimate the product $\mathbf{X}_{\text{band}}\mathbf{Y}_{\text{DD}}$ by selecting which $\mathbf{Y}_m$ participate on the sum (11). Since this approach aims to reduce the computational cost, one must select the data before operating the outer products. Therefore, we propose five approaches to select $L < M$ elements for the estimate.

- **i) Fully Randomized (rand):** In this approach, the probability of selecting a row from $\mathbf{Y}_{\text{DD}}$ follows a uniform distribution in a pooling with sample replacement, i.e., a row can be selected more than once and the probability is always $p_m = 1/M$.
- **ii) Randomized ℓ_1 (rand-ℓ_1):** Here, the probability of selecting a row is shaped by the ℓ_1-norm, i.e., the probability of each row is $p_m = \frac{||\mathbf{y}_{\text{DD}}(m)||_1}{\sum_{m=0}^{M-1} ||\mathbf{y}_{\text{DD}}(m)||_1}$; the pooling also has sample replacement.
- **iii) Randomized ℓ_2 (rand-ℓ_2):** This method follows the same principles of **rand**-ℓ_1, however, we replace the ℓ_1-norm by the ℓ_2-norm.
- **iv) Deterministic ℓ_1 (det-ℓ_1):** In this method we select the L rows with the highest values for the ℓ_1-norm.
- **v) Deterministic ℓ_2 (det-ℓ_2):** Similarly to **det**-ℓ_1, we select L rows based on the ℓ_2-norm.

The rand-ℓ_2 method is motivated by [6], where the norm shapes the probability density function (PDF) for the randomized sampling; it induces the sampler to more often choose rows with more energy. We propose using the norm-ℓ_1 as a method to further reduce the computational burden and maintain a similar PDF characteristic. On the other hand, the fully randomized method is an approach that ignores any information from the rows, making their selection equiprobable. The deterministic methods are naturally motivated by questioning how a deterministic approach would perform compared to the randomized approaches.

Furthermore, reweighting is necessary to ensure proper approximation [6], [7], and for the radar operation where a Neyman-Pearson detector is commonly applied [15], [16]. From [7], the weighting factor of each $\mathbf{Y}_m$ term in (11) should be $1/(p_m L)$ for the randomized matrix product approximation. In the case of the deterministic approach, we use the same weighting factor, where p_m is estimated using the norm of the m-th row. Hence, let $\mathcal{L}$ be the set (with possible repetitions) of selected elements, the approximation is calculated as

$$\tilde{\mathbf{Y}}_{\text{des}} = \sum_{m \in \mathcal{L}} \frac{\mathbf{Y}_m}{p_m L}. \quad (12)$$

B. Target Detection

Let H_1 be the hypothesis that a target is present, H_0 its counterpart, we have

$$\frac{|\mathbf{Y}_{\text{des}}(i,j)|}{D} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \quad (13)$$

where $\mathbf{Y}_{\text{des}}(i,j)$ is the element of the i-th row and the j-th column from $\mathbf{Y}_{\text{des}}$ and D is a normalization factor that depends on the sequence used in $\mathbf{x}(0)$; in the case of ZC sequences, D equals the length of the sequence.

IV. Results

We simulated a sensing experiment using the proposed system with different values for the number of elements in the approximation, $0 < L < M$. The communication system is configured for a carrier frequency $f_c = 4$ GHz, with $M = 128$ subcarriers evenly spaced by $\Delta f = 15$ kHz and an OTFS frame consisting of $N = 32$ blocks. The environment has $P = 10$ targets, randomly generated with velocity $v_p \in \left[-c\frac{\Delta f}{2f_c}; c\frac{\Delta f(N-1)}{2Nf_c}\right]$ m/s, and range $d_p \in \left[0; \frac{c}{2\Delta f}\right]$ m, where c is the constant speed of light. The experiment is carried out in a Monte Carlo process, with 6000 samples, to estimate the probability of detection P_d and the probability of false alarm P_f given different threshold values $0 < \gamma \leq 1$, and signal-to-noise ratio (SNR). We use recall as an estimate for P_d and false positive rate for P_f,

$$P_d = \frac{\text{TP}}{\text{TP} + \text{FN}}, \qquad P_f = \frac{\text{FP}}{\text{FP} + \text{TN}}, \quad (14)$$

where TP, FN, FP, and TN are the true positive, false negative, false positive, and true negative counts, respectively. Concerning the transmitted sequence $\mathbf{x}(0)$, we explore two scenarios, one with full dynamic, where we use a ZC sequence of length M and the other with half dynamic, where the sequence has $M/2$ elements and $\mathbf{x}(0)$ is completed with zero padding.

In Fig. 1, we present the so-called receiver operation curve (ROC) for the different systems, with $L = 16$ and SNR $= 0$ dB and full dynamic. Here the approximate systems are compared to one without approximation, indicated as "Complete".

From Fig. 1, it is clear that despite the reduced number of elements in the approximation, the systems are capable of identifying targets with performance comparable to a system without approximation; however, for smaller values of P_f,

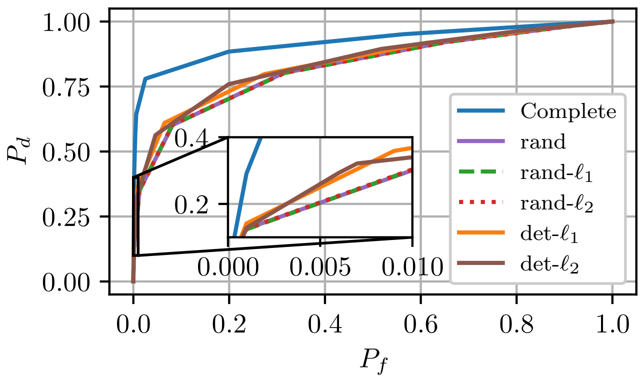

Fig. 1. ROC for the different approximation systems, using $L = 16$ and for an SNR = 0 dB and full dynamic sequence.

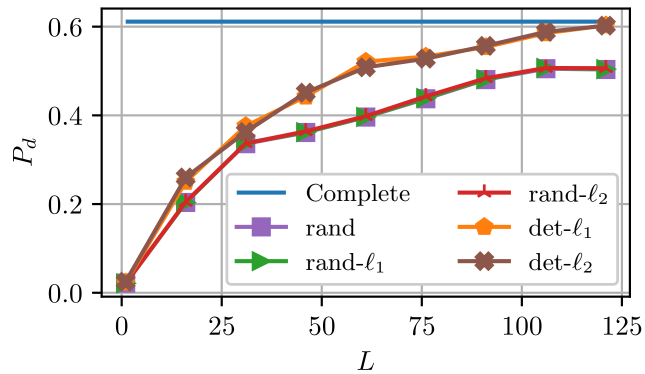

Fig. 3. Probability of target detection P_d, per number of elements in the approximation L, with SNR = 0 dB and $P_f = 0.5\%$.

the detection probability degrades slightly using a coarse approximation.

In Fig. 2, we compare the probability of detection of the proposed approximations as a function of SNR considering $L = 16$, $P_f = 0.5\%$ and full dynamic. One observes that all systems with the approximation have degraded performance when compared to the Complete system. Furthermore, we observe a slight increase in performance for the systems using a deterministic approach if compared to their randomized counterparts.

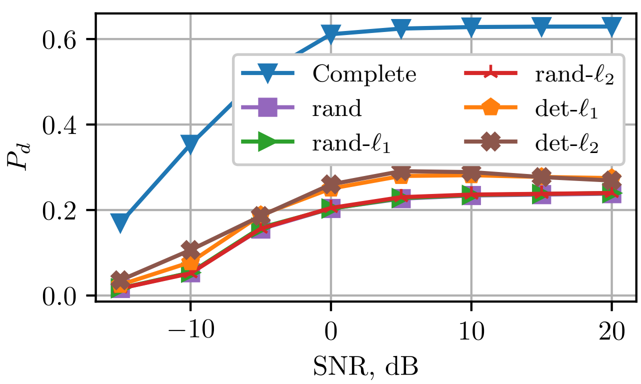

Fig. 2. Probability of target detection, P_d, per SNR level for systems with $P_f = 0.5\%$ and approximations with $L = 16$ and full dynamic.

Fig. 3 presents the curves of P_d for values of L in a scenario of SNR = 0 dB and constant $P_f = 0.5\%$ and full dynamic. We observe an increase in P_d for increments in L, it is also noticeable a slightly better performance in systems using the deterministic approach in contrast to the ones with randomized selection. Moreover, the randomized systems are incapable of reaching the same performance as the Complete and deterministic systems for high values of L; we credit this behavior to the pooling with sample replacement in these randomized systems, where the number of selected elements is smaller than L.

Lastly, in Fig. 4, we present the curves of P_d for values of L for sequences using half the dynamic, where the sequence occupies only half of the $M = 256$ samples; we also set the SNR = 0 dB and constant $P_f = 0.5\%$. We observe that all approximation systems reach a plateau at approximately $L = 128$ which is the length of the transmitted sequence. We also observe that the system det-ℓ_2 decreases in performance as L increases for $L > 128$, this is due to the reweight that generates errors in the approximation.

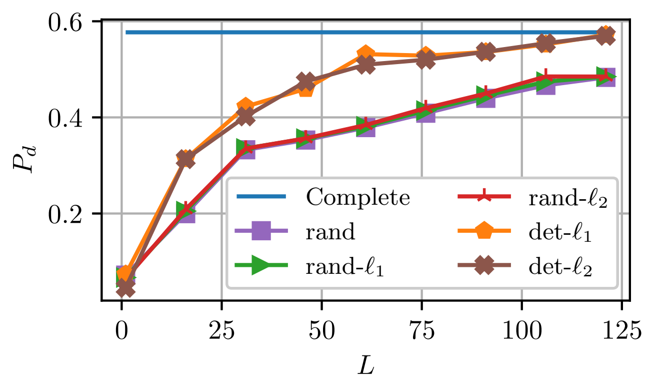

Fig. 4. Probability of target detection P_d, per number of elements in the approximation L using half dynamic, with SNR = 0 dB and $P_f = 0.5\%$.

A. Complexity Analysis

Let us analyze the computational cost of (11), which is the computation we want to approximate. Without any approximation, it requires M^2N scalar multiplications and M^2N sums, resulting in a $\mathcal{O}(M^2N)$ computational complexity. By considering an approximation using $L < M$ elements, the computation is reduced to $(L+1)MN$ multiplications if we consider the reweight and LMN additions entailing a computation cost of $\mathcal{O}((L+1)MN)$.

The randomized algorithms might further reduce the complexity by selecting the same elements multiple times. However, the norm calculation should be considered in some propositions. The ℓ_1-norm is a simple sum of the absolute value of the elements in a row, which results in MN additions, and the ℓ_2-norm takes MN multiplications and N additions. In the case of the deterministic methods rely on sorting algorithms that take up to $\mathcal{O}(M^2)$ in complexity [17].

Moreover, random sampling following a probability distribution may have a considerable computational cost, as the algorithm described in [18], it takes up to LM multiplications to perform this task, including an additional $\mathcal{O}(LM)$ to the method's complexity.

TABLE I
ESTIMATED COMPLEXITY OF EACH METHOD, FOLLOWED BY MAXIMAL VALUE OF L FOR ACHIEVING COMPLEXITY REDUCTION, NOTED AS L^*.

Method	**Complexity**	L^*
Complete	$\mathcal{O}(M^2N)$	$-$
rand	$\mathcal{O}((L+1)MN)$	$M-1$
rand-ℓ_1	$\mathcal{O}((L+1)MN+LM)$	$\frac{(M-1)N}{N+1}$
rand-ℓ_2	$\mathcal{O}((L+2)MN+LM)$	$\frac{(M-2)N}{N+1}$
det-ℓ_1	$\mathcal{O}((L+1)MN+M^2)$	$M-1-\frac{M}{N}$
det-ℓ_2	$\mathcal{O}((L+2)MN+M^2)$	$M-2-\frac{M}{N}$

Table I summarizes the estimated complexity for each method. In the sense of computational complexity, the randomized algorithms outperform their deterministic counterparts as the sorting introduces greater complexity than sampling a value following an arbitrary probability distribution; this difference shortens as L approximates M. The proposed algorithms attain a complexity reduction, in comparison to the Complete product, as long as $L < L^*$.

V. CONCLUSION

This work proposed a strategy to reduce the computational cost of sensing in joint communication and sensing systems. The solution utilizes innovative rank-reduction tools to reduce the dimension of the correlation matrix required to perform localization tasks within the ISAC framework. The presented simulations corroborate that the contribution here is feasible for practical implementations.

The results show that the additional cost of using any of the norms to model the probability of term selection in randomized approximations brings negligible gain. However, the norm information adds approximately 10% to the target detection probability in a deterministic proposition when compared to the randomized methods. This performance increment is achieved at the cost of computing the ℓ_1-norm for each row and a sorting algorithm, approximately an $\mathcal{O}(M^2)$ surplus.

The proposed method brings a clear tradeoff between the amount of computation and the probability of target detection in the DD domain. The simulation was performed using Python, and the code is available at https://github.com/felipescoelho/approximation-otfs-sensing.

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