Bayesian distance estimation with incoherent multi-frequency two-way phase measurements

Kimmo Kansanen, Jenny Aune Forbord, Johan Suárez
Department of Electronic Systems, Norwegian University of Science and Technology, Trondheim, Norway
kimmo.kansanen@ntnu.no, jenny.forbord@gmail.com, johan.suarez@ntnu.no

***Abstract*—Two-way phase measurements over multiple frequencies can be used to measure the distance between two radios. An unknown phase offset between the two radios on each measured frequency must be accommodated by the estimator. We derive a maximum a-posteriori estimator for the distance in a channel with a single propagation path from first principles by treating the unknown phase as a nuisance parameter. We develop the Modified Cramer-Rao Lower Bound for the estimator and verify the estimator reaches it. We compare the estimator against state of the art estimators and their performance bounds. Under high signal-to-noise ratio the proposed estimator outperforms all known algorithms by 6dB for Bluetooth Low Energy system parameters.**

***Index Terms*—Bluetooth Low Energy (BLE), distance estimation, ranging, localization, phase-measurement, Bayesian estimation**

I. Introduction

Positioning has become an integrated part of many wireless systems due to the demand from many new applications such as indoor navigation, asset tracking and autonomous vehicle navigation. A solution integrated with the wireless often reduces the required number of subsystems, while enabling the wireless system itself to take advantage of the position information. Bluetooth Low Energy (BLE) has widespread adoption due to its affordability and low-power consumption, and has emerged as a key enabler for proximity-based services and short distance positioning solutions.

Recent advances in BLE-based distance estimation have demonstrated that Phase Difference of Arrival (PDoA) [1] can provide superior performance compared to traditional methods relying on Received Signal Strength Indicator (RSSI) [2], Time of Arrival (ToA) [3] or Time Difference of Arrival (TDoA) estimation [4].

In a free-space scenario, the phase rotation of the propagated wave is directly proportional to both the frequency and distance traveled. The Multi-Carrier Phase Difference (MCPD) algorithm utilizes two narrowband transceivers that measure this phase shift at two (or more) predetermined frequencies and estimates the distance between the transceivers based on these[1]. In [5] it was shown that the approach is compatible with BLE hardware. In [6] the Fast Fourier Transform (FFT) of the Power Spectral Density (PSD) of the signal's phase was utilized to obtain an improved error performance that reaches the relevant Cramer Rao Bound.

Other related works that consider a two way signal model show different approaches to multipath environments. A solution based on the MUSIC estimator [7] can balance between accuracy and resolving multipath components at the cost of being computationally intensive. In [8] an enhanced MUSIC algorithm and a sparse Orthogonal Matching Pursuit (OMP) algorithm are shown, focusing on lower complexity and improved performance. The authors of [9] employ a Support Vector Regression (SVR) approach that outperform MUSIC in both accuracy and computational efficiency and achieves decimeter-level precision across diverse scenarios. The authors of [10] consider narrowband systems with missing or interfered tones and apply atomic norm minimization and a Neural Network (NN).

One commonality of all the prior work above is their treatment of the unknown phase component present in the measurements due to the phase and clock offset between the two transceivers. The works above eliminate this phase uncertainty by adding the phases measured by the two transceivers on each frequency, thereby canceling the unknown offset.

Our contribution is based on treating the unknown phase offset between the transceivers as a nuisance parameter in the Bayesian framework. The system and observation model follows that in [5]. We build on the work within [11], where the expressions for the estimator were first developed. We focus on the scenario of a single-path propagation channel and derive the Bayesian maximum a-posteriori (MAP) estimator for distance from first principles. We develop a simple gradient search based approach to find the MAP estimate. We derive the Modified Cramer-Rao Bound (MCRB) for the problem and verify that the estimator approaches the bound at high Signal to Noise Ratio (SNR). Numerical comparison shows the estimator outperforms all known algorithms by 6dB at high SNR, while at medium SNR the performance depends on the search initialization.

II. System model

Two radio transceiver units, denoted the Initiator and Reflector, stand r meters apart, and thereby the phase rotation of a sinusoidal electromagnetic signal at frequency f, due to the time of flight, is given by

$$\phi(f, r) = -2\pi f r / c_0 \pmod{2\pi}, \tag{1}$$

where c_0 denotes the speed of light.

Our system utilizes a total of K_f frequencies on a regular grid, separated by Δ_fHz, indexed by $k \in \{1 \dots K_f\}$. As all the measurements are performed on the baseband, the frequencies equal $f_k = k\Delta_f$. The transceiver units have coarsely synchronized clocks, such that they run at identical frequencies but with epochs misaligned with Δ_t seconds when the measurement on a frequency is performed. There exists also an unknown phase difference between the two oscillators. When the Initiator sends a sinusoidal at frequency f_k the phase at the Reflector is given by

$$\phi_R(f_k, r) = -2\pi f_k \left(\frac{r}{c_0} - \Delta_t\right) + \theta_k \pmod{2\pi} \quad (2)$$

$$\phi_R(f_k, r) = -2\pi f_k \left(\frac{r}{c_0}\right) + \Delta\phi_k \pmod{2\pi}, \quad (3)$$

where we have included both clock and local oscillator phase differences into one phase term $\Delta\phi_k$. Respectively, when the Reflector sends back a sinusoid to the Initiator, and assuming the oscillators and clocks are stable enough for θ_k and Δ_t to remain constant over the measurements, the measured phase at Initiator equals

$$\phi_I(f, r) = -2\pi f_k \left(\frac{r}{c_0}\right) - \Delta\phi_k \pmod{2\pi}. \quad (4)$$

Note that $\Delta\phi_k$ is a common term in (3)-(4), although with opposite sign. It is assumed that the oscillators cannot maintain phase control during frequency change, and thereby $\Delta\phi_k$ is independent for different frequencies.

With the signal amplitudes at Reflector and Initiator, denoted as a_R and a_I respectively, we can state the model for the complex signals at Reflector and Initiator, excluding any observation noise, as

$$x_R(f_k, r) = a_R e^{j(-2\pi f_k(r/c_0)+\Delta\phi_k)} \quad (5)$$

$$x_I(f_k, r) = a_I e^{j(-2\pi f_k(r/c_0)-\Delta\phi_k)}. \quad (6)$$

The Initiator and Reflector perform a measurement on each frequency, and we denote the observed values as

$$\tilde{y}_R(f_k) = a_{R,k} e^{j\theta_{R,k}} = x_R(f_k, r) + w_R(f_k) \quad (7)$$

$$\tilde{y}_I(f_k) = a_{I,k} e^{j\theta_{I,k}} = x_I(f_k, r) + w_I(f_k). \quad (8)$$

In the above, $w_R(f_k)$ and $w_I(f_k)$ denote the white Gaussian noise at Reflector and Initiator, respectively. The noise has variance σ_0^2 and is assumed i.i.d. between the Reflector and Initiator, and across different frequencies f_k.

The measurements from Reflector are transmitted to the Initiator for further processing. For the development of the estimator, we simply use the shorthand $\mathbf{y}_k = (\tilde{y}_R(f_k), \tilde{y}_I(f_k))$ for the pair of measurements on frequency f_k. For a random variable x with distribution $p(x)$, we denote $\mathbb{E}_x[g(x)] = \int g(x)p(x)dx$.

III. Maximum A-Posteriori estimator for distance

In this section we develop a Maximum A-Posteriori (MAP) estimator for distance. We begin by expressing the general form of the Bayesian multi-frequency estimator for our problem, and then develop the detailed estimator expressions for the observation model in Eqs. (5)-(6).

A. Bayesian estimator form for two-way measurements: general structure

For simplicity, we present the estimator for $K_f = 2$, and the general form for arbitrary K_f follows directly. The distance r has the joint distribution with the observations and nuisance parameters given by

$$\begin{aligned} &p(r, \mathbf{y_1}, \mathbf{y_2}, \Delta\phi_1, \Delta\phi_2) \\ &= p(r, \Delta\phi_1, \Delta\phi_2 | \mathbf{y_1}, \mathbf{y_2}) p(\mathbf{y_1}, \mathbf{y_2}) \\ &= p(\mathbf{y_1}, \mathbf{y_2} | \Delta\phi_1, \Delta\phi_2, r) p(r) p(\Delta\phi_1) p(\Delta\phi_2). \end{aligned} \quad (9)$$

We can then express the maximum a-posteriori estimator for r via marginalization over the nuisance parameters $\Delta\phi_1$ and $\Delta\phi_2$ as

$$\hat{r} = \arg\max_r p(r|\mathbf{y_1}, \mathbf{y_2}) \quad (10)$$

$$= \arg\max_r \mathbb{E}_{\Delta\phi_1, \Delta\phi_2}\left[p(r, \Delta\phi_1, \Delta\phi_2 | \mathbf{y_1}, \mathbf{y_2})\right]. \quad (11)$$

Applying the two decompositions of the joint distributions in (9), we can express (11) as

$$\hat{r} = \arg\max_r \mathbb{E}_{\Delta\phi_1, \Delta\phi_2}\left[\frac{p(\mathbf{y_1}, \mathbf{y_2}|\Delta\phi_1, \Delta\phi_2, r)p(r)}{p(\mathbf{y_1}, \mathbf{y_2})}\right]. \quad (12)$$

Considering that the denominator does not depend on r, and that the observations, conditioned on r, as well as the nuisance parameters are independent across frequencies, we can simplify (12) to

$$\begin{aligned} \hat{r} &= \arg\max_r \mathbb{E}_{\Delta\phi_1, \Delta\phi_2}\left[p(\mathbf{y_1}|\Delta\phi_1, r)p(\mathbf{y_2}|\Delta\phi_2, r)p(r)\right] \\ &= \arg\max_r\, p(r) \\ &\quad \times \mathbb{E}_{\Delta\phi_1}\left[p(\mathbf{y_1}|\Delta\phi_1, r)\right]\mathbb{E}_{\Delta\phi_2}\left[p(\mathbf{y_2}|\Delta\phi_2, r)\right]. \end{aligned} \quad (13)$$

We can now state the generic form for the estimator for any K_f as a straightforward extension of (13) as

$$\hat{r} = \arg\max_r p(r)\Pi_{k=1}^{K_f}\mathbb{E}_{\Delta\phi_k}\left[p(\mathbf{y}_k|\Delta\phi_k, r)\right]. \quad (14)$$

B. Bayesian estimator form for two-way measurements: detailed derivation

Here, we develop the detailed expressions of the distance estimator (14) for our signal model (7)-(8). We explicitly assume that the estimator has perfect knowledge of the amplitude, and that the channel amplitude behaves reciprocically, i.e. $a_R = a_I = a_{R,k} = a_{I,k}$. We first observe that

$$\begin{aligned} p(\mathbf{y}_k|\Delta\phi_k, r) &= p(\tilde{y}_R(f_k), \tilde{y}_I(f_k)|\Delta\phi_k, r) \\ &= p(\tilde{y}_R(f_k)|r, \Delta\phi_k)p(\tilde{y}_I(f_k)|r, \Delta\phi_k) \end{aligned} \quad (15)$$

since, conditioned on r and $\Delta\phi_k$, the measurements $\tilde{y}_R(f_k)$ and $\tilde{y}_I(f_k)$ are independent due to the independence of noise terms in Eqs. (7) and (8).

The measurement $\tilde{y}_R(f_k)$ is a complex Gaussian with distribution [11, Eq. (2.31)]

$$p(\tilde{y}_R(f_k)|r,\Delta\phi_k) = \\ = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{|\tilde{y}_R(f_k)|^2 + |a_k|^2}{2\sigma_0^2}\right\} \\ \times \exp\left\{\frac{2|a_k|^2 \cos\left(-\frac{2\pi f_k r}{c_0} + \Delta\phi_k - \theta_{R,k}\right)}{2\sigma_0^2}\right\}. \quad (16)$$

Respectively, the measurement $\tilde{y}_I(f_k)$ is a complex Gaussian with distribution [11, Eq. (2.32)]

$$p(\tilde{y}_I(f_k)|r,\Delta\phi_k) = \\ = \frac{1}{\sqrt{2\pi\sigma_0^2}} \times \exp\left\{-\frac{|\tilde{y}_I(f_k)|^2 + |a_k|^2}{2\sigma_0^2}\right\} \\ \times \exp\left\{\frac{2|a_k|^2 \cos\left(-\frac{2\pi f_k r}{c_0} - \Delta\phi_k - \theta_{I,k}\right)}{2\sigma_0^2}\right\}. \quad (17)$$

Finally, we need the a-priori distribution for the phase uncertainty $\Delta\phi_k$. As this is the result of time and phase offset between the Initiator and Reflector hardware, it is reasonable to assume maximum uncertainty and use the uniform distribution, and thereby we have

$$p(\Delta\phi_k) = \frac{1}{2\pi} \quad \forall k.$$

Now, using the fact that $\cos(a) + \cos(b) = 2\cos((a+b)/2)\cos((a-b)/2)$, we can express Eq. (15) as [11, Eq. (2.34)]

$$p(\mathbf{y}_k|\Delta\phi_k, r) \\ = \frac{1}{4\pi^2\sigma_0^2} \exp\left\{-\frac{|\tilde{y}_R(f_k)|^2 + 2|a_k|^2 + |\tilde{y}_I(f_k)|^2}{2\sigma_0^2}\right\} \\ \times \exp\left\{\frac{2|a_k|^2}{\sigma_0^2}\left(\cos\left(-\frac{2\pi f_k r}{c_0} - \frac{\theta_{R,k}+\theta_{I,k}}{2}\right)\right.\right. \\ \left.\left. \times \cos\left(\Delta\phi_k - \frac{\theta_{R,k}-\theta_{I,k}}{2}\right)\right)\right\}. \quad (18)$$

Finally, we can integrate over $\Delta\phi_k$ by [12] [11, Eq. (A.24)] to obtain

$$E_{\Delta\phi_k}\left[p(\mathbf{y}_k|\Delta\phi_k, r)\right] \\ = \frac{1}{2\pi^2\sigma_0^2} \exp\left\{-\frac{|\tilde{y}_R(f_k)|^2 + 2|a_k|^2 + |\tilde{y}_I(f_k)|^2}{2\sigma_0^2}\right\} \\ \times I_0\left(\frac{2|a_k|^2}{\sigma_0^2} \cdot \cos\left(-\frac{2\pi f_k r}{c_0} - \frac{\theta_{R,k}+\theta_{I,k}}{2}\right)\right), \quad (19)$$

where $I_0(\cdot)$ denotes the modified Bessel function of the first kind. We can then express the MAP distance estimator as the product over all frequencies

$$\hat{r} = \arg\max_r p(r)\Pi_{k=1}^{K_f} p(\mathbf{y}_k|r) \\ = \arg\max_r p(r)\Pi_{k=1}^{K_f} \\ I_0\left(2\gamma\cos\left(-\frac{2\pi f_k r}{c_0} - \frac{\theta_{R,k}+\theta_{I,k}}{2}\right)\right), \quad (20)$$

where we have also removed all constants from the product, and defined the SNR as $\frac{|a_k|^2}{\sigma_0^2} = \gamma$. Alternatively, we can express the estimator as

$$\hat{r} = \arg\max_r M(r), \quad (21)$$

where the *a-posteriori* metric is given in the numerically more interesting form of

$$M(r) = \log p(r) + \\ \sum_{k=1}^{K_f} \log I_0\left(2\gamma\cos\left(-\frac{2\pi f_k r}{c_0} - \frac{\theta_{R,k}+\theta_{I,k}}{2}\right)\right) \quad (22)$$

Computing a value for this estimation metric, in order to find the maximum, requires the knowledge of $a_I = a_R$. In the single path case considered here, the amplitude is fully correlated over frequencies and we can estimate the amplitude from the measurements with a high accuracy, especially if K_f is large.

In the rest of the paper we concentrate on the estimator without the the a-priori information, as including the latter is straightforward.

IV. Metric Shape: Bluetooth Low Energy Example

Bluetooth Low Energy can perform the two-way measurements as described in Section II. The system parameters relevant to the algorithm are listed in Table I.

TABLE I
BLE PARAMETERS

Frequency range	2402–2480MHz
Number of carriers K_f	77
Carrier spacing Δ_f	1MHz

We reproduce the metric (22) for γ =10dB in Figure 1 for the special case when $a(f_k) = a_R(f_k) = a_I(f_k)$ and $\theta_{R,k} = \theta_{I,k} = 2\pi f_k(r/c_0)$, i.e. under noise-free observations.

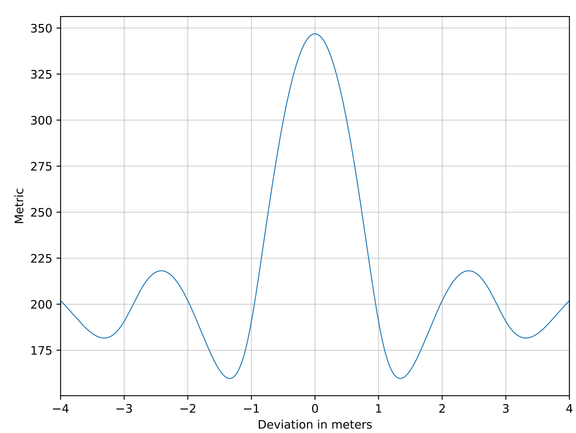

Fig. 1. A-Posteriori metric shape, centered around the correct distance

The experiment reveals that the metric exhibits multiple local maxima in addition to the global one, with the two nearest local minima at approximately 2.4 meters. This implies that a line search will converge to a local maximum depending on the initialisation.

V. Line Search for Estimate

We can initialize the search with any of the known algorithms in e.g. [1][6] to obtain an initial estimate r_0. A simple gradient iteration would then find in each iteration i the new estimate

$$r_{i+1} = r_i - \delta_s \frac{d}{dr} M(r).$$

for some step size $\delta_s > 0$. We can terminate the iteration when the gradient $\frac{d}{dr}M(r)$ reaches some lower limit δ_M. Let us denote

$$\psi(r) = -\frac{2\pi f_k r}{c_0} - \frac{\theta_{R,k} + \theta_{I,k}}{2}$$

to be able to compactly express the gradient as

$$\frac{d}{dr} M(r) = -\frac{\frac{2\pi f_k}{c_0} \gamma I_1 \left(\gamma \cos(\psi(r))\right) \sin(\psi(r))}{I_0 \left(\gamma \cos(\psi(r))\right)}.$$

By setting manually the step size to $\delta_s = 0.01$ and terminating the iteration with $\delta_M = 10^{-3}$, the algorithm usually finishes in under 10 iterations at signal to noise ratios above 5dB. By computing the metric (22) at ± 2.4 meters distance from the gradient search result we can test if the result is a local maximum. We can then re-initialize the search with an improved initial estimate if this is the case. We denote the approach the Local-to-Global Maximum Search (LGMS).

VI. Performance Bounds

In the case of estimation in the presence of nuisance parameters the variance of any unbiased estimator is bounded by the MCRB [13]. We derive the MCRB in Appendix A following the approach in [14] as

$$\text{MCRB} = \frac{3c_0^2}{8\gamma\pi^2\Delta_f^2 K_f(K_f+1)(2K_f+1)}. \quad (23)$$

The Cramer-Rao Lower Bound (CRLB) provided in [15], [16] is the relevant lower bound for our benchmark algorithms [1][5][6], and is given by

$$\text{CRLB} = \frac{3c_0^2}{4\gamma\pi^2\Delta_f^2 K_f \left(K_f^2 - 1\right)}. \quad (24)$$

The ratio of (24) and (23) can be derived to equal

$$\frac{2(2K_f+1)}{K_f-1} \approx 4$$

where the approximation is tight for large K_f. This implies a gain by a factor of four in MSE, or a 6dB gain in SNR.

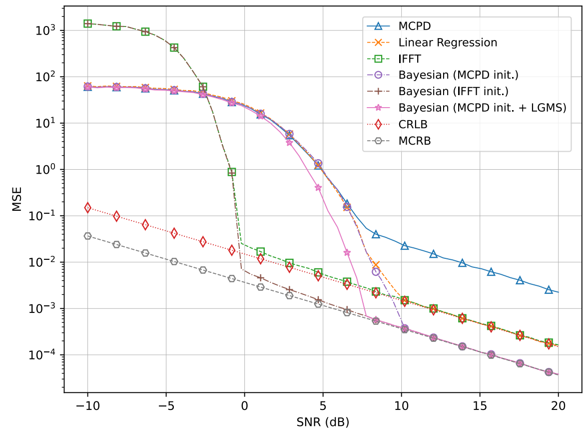

Fig. 2. MSE comparison to known algorithms and bounds

VII. Numerical Examples

In this section, the behaviour of the Bayesian estimator is tested with simulations utilizing the BLE parameters given in Table I. We compare the proposed estimator to state of the art algorithms in terms of Mean Square Error (MSE), shown in Figure 2.

The benchmark algorithms include MCPD, Linear Regression, and the IFFT approach. The MCPD algorithm estimates the distance via the mean of phase differences between pairs frequencies [1][5]. The linear regression fits a line on the unwrapped phase of all the frequencies [1]. The IFFT algorithm performs an Inverse Fast Fourier Transform on the Power Spectral Density (PSD) of the signal's phase [6]. The relevant lower bound for these estimators is the CRLB. The proposed estimator is initialized with either the MCPD or the IFFT estimate, and then the search detailed in Section V is performed (without LGMS). The performence of the MCPD initialised estimator with the LGMS post-processing is also reported.

The results indicate that the proposed estimator reaches the MCRB at high SNR, regardless of the initialization. This implies it outperforms all currently known algorithms at high SNR by 6dB.

The superior performance of the IFFT based initialization allows the estimator to reach the MCRB at approximately 6dB lower SNR than with the MCPD initialization. The MCPD initialization often leads to iterations converging to one of the local maxima closest to the global one. As seen in Figure 2, the LGMS approach, designed to mitigate the issue, is able to outperform the plain MCPD initialized algorithm, even though not reaching the accuracy of the IFFT initialized algorithm.

VIII. Conclusions

The signals from the two-way measurements are conditionally independent given the distance, and are only coupled within each frequency by the one unknown phase parameter.

The Bayesian approach for treating these nuisance parameters results in an a-posteriori estimation metric that is numerically easy to deal with, and a simple gradient search finds a maximum in a limited number of iterations when the initialisation is sufficiently precise. Avoiding local maxima of the metric at medium SNR requires sufficient initialisation precision, e.g. via the IFFT based algorithm[6]. The estimator reaches the Modified Cramer Rao Bound at high SNR and outperforms currently known estimators by 6dB.

Appendix A
MCRB Derivation

The observation vector $\mathbf{y} = \left(y_1 \cdots y_{K_f}\right)^T$ can be expressed as

$$\mathbf{y} = \mathbf{s}(r, \mathbf{u}) + \mathbf{w}, \tag{25}$$

where $\mathbf{s} = (r, \mathbf{u})$ is a vector of the noiseless observations, $\mathbf{u} = \left(\Delta\phi_1 \cdots \Delta\phi_{K_f}\right)^T$ is the vector of nuisance parameters, and $\mathbf{w}$ is a white Gaussian noise vector with zero mean and variance σ_0^2. The vector $\mathbf{s} = \left(s_1 \cdots s_{K_f}\right)^T$ considers the noiseless observation from Eqs. (5)-(6) rewritten in their trigonometric form as pairs of real-valued quantities:

$$s_k(\mathbf{r}, \mathbf{u}) = \begin{bmatrix} a_k \cos\left(2\pi f_k \frac{r}{c_0} \pm \Delta\phi_k\right) \\ -a_k \sin\left(2\pi f_k \frac{r}{c_0} \pm \Delta\phi_k\right) \end{bmatrix} \tag{26}$$

The MCRB can be defined as [14]

$$\mathrm{MCRB}(r) = \frac{1}{2\gamma \mathbb{E}_{\mathbf{u}}\left[\left|\frac{\partial}{\partial r}\mathbf{s}(r, \mathbf{u})\right|^2\right]} \tag{27}$$

The squared norm of the partial derivatives is expressed as

$$\begin{aligned} \left|\frac{\partial \mathbf{s}(r, \mathbf{u})}{\partial r}\right|^2 &= \sum_{k=1}^{K_f} \left(\frac{\partial s_k(r, \mathbf{u})}{\partial r}\right)^2 \\ &= \sum_{k=1}^{K_f} \left(\frac{4\pi^2 \Delta_f^2 k^2 a^2}{c_0^2}\right) \left(\sin^2\left(\frac{2\pi\Delta_f kr}{c_0} + \Delta\phi_k\right)\right. \\ &+ \cos^2\left(\frac{2\pi\Delta_f kr}{c_0} + \Delta\phi_k\right) + \sin^2\left(\frac{2\pi\Delta_f kr}{c_0} - \Delta\phi_k\right) \\ &+ \left.\cos^2\left(\frac{2\pi\Delta_f kr}{c_0} - \Delta\phi_k\right)\right). \end{aligned} \tag{28}$$

Using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, the expression reduces to

$$\left|\frac{\partial \mathbf{s}(r, \mathbf{u})}{\partial r}\right|^2 = 2\sum_{k=1}^{K_f} \left(\frac{4\pi^2 \Delta_f^2 k^2 a^2}{c_0^2}\right) = \sum_{k=1}^{K_f} \frac{8\pi^2 \Delta_f^2 k^2 a^2}{c_0^2}. \tag{29}$$

The expectation with respect to $\mathbf{u}$ is not needed and the sum only concerns one variable, so the MCRB is

$$\mathrm{MCRB}(r) = \frac{c_0^2}{16\gamma\pi^2\Delta_f^2 a^2} \sum_{k=1}^{K_f} k^2. \tag{30}$$

We can now use the sum of squares formula, and after simplification the expression for the MCRB results in (23).

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