ULA-inspired Two-Fold Redundant Sparse Linear Array With Closed-Form Expressions and 2/N Fragility

Ashish Patwari
School of Electronics Engineering,
Vellore Institute of Technology, Vellore
Vellore, Tamil Nadu, India
*ashish.p@vit.ac.in
https://orcid.org/0000-0001-9489-7004

Priyadarshini Raiguru
Department of ECE, ITER
SOA Deemed to be University
Bhubaneswar, Odisha, India
priyadarshiniraiguru@soa.ac.in
https://orcid.org/0000-0002-9504-0720

Ashok Chandrasekaran
School of Computing and Data Science,
Sai University,
Chennai, Tamil Nadu, India
ashok.c@saiuniversity.edu.in
https://orcid.org/0000-0002-6292-8643

***Abstract*—Many existing sparse arrays are maximally economical in that they retain the fewest number of sensors required to realize a given aperture while maintaining a hole-free coarray. As a result, they cannot tolerate the failure of even a single sensor. Although coarray interpolation techniques circumvent coarray discontinuities and provide accurate direction of arrival (DOA) estimation, their computational complexity is prohibitive for use in real-time systems. In contrast, two-fold redundant sparse arrays (TFRSAs) have inherent redundancy to tackle single sensor failures. Finding novel TFRSAs has remained an unsolved challenge in sparse array literature, mainly because of the NP-hard nature of the optimization problem. Existing TFRSA designs either lack closed-form expressions or consist of hidden dependencies or are valid only for a few array sizes. We propose a simple approach to find TFRSAs. This formulation is valid for all array sizes ($N \geq 6$), provides closed-form expressions for sensor positions, offers a fragility of exactly $2/N$ as expected of TFRSAs, and provides $O(N)$ degrees of freedom (DOFs) as required for next-generation wireless systems. Detailed proofs for two-fold redundancy in the healthy case and coarray continuity during sensor failures are provided. MATLAB simulations reinforce the correctness of the proposed formulations.**

***Keywords*—Array signal Processing, Difference coarray, Sensor Failures, Spatial Lags, Sparse arrays, Two-fold redundant arrays, Weight function.**

I. Introduction

Sensor arrays find numerous applications in fields such as radar, sonar, acoustics, medical imaging, and wireless communications. A sensor array consists of two or more sensors arranged in a particular geometry such as linear, planar, circular, or spherical. In most applications, the array must determine the directions of incoming wave fields. Sparse arrays are preferable to uniform arrays in more than one way. The former have been extensively studied in the past 15 years. The coprime array and the two-level nested array, introduced in 2010, provided closed-form expressions (CFEs) to determine sensor positions, breaking the earlier reliance on exhaustive search techniques. Since then, the field of sparse array design has experienced extraordinary growth. A thorough review of the types and properties of sparse arrays was presented in [1], [2], [3]. Invaluable insights into sensor arrays and their applications can be found in [4], [5], [6].

Sparse arrays such as coprime arrays, nested arrays, and their variants offer superior direction estimation performance compared to uniform linear arrays (ULAs) for the same number of sensors. This advantage is attributed to the larger apertures, higher degrees of freedom (DOFs), enhanced angular resolution, and reduced mutual coupling possessed by the former. Most sparse arrays are designed to be maximally economic in terms of coarray/spatial redundancy, rendering them vulnerable to the failure of even a single sensor. Enhancing coarray redundancy through the use of two-fold redundant sparse arrays (TFRSAs) can increase the array's robustness against such failures. Nevertheless, the discovery of novel TFRSAs remained a significant challenge in the sparse array literature, primarily owing to the NP-hard nature of the problem, which involves finding optimal sensor positions that satisfy diverse constraints. The design of novel TFRSAs was extensively pursued during 2017-2020 to determine sparse array architectures with an inherent tolerance to sensor failures. Many TFRSA formulations have been proposed in the existing literature [7], [8], [9], [10], [11].

A. Existing methods and gaps in the literature

Liu and Vaidyanathan envisaged finding robust minimum redundant arrays (RMRAs) that provide the largest aperture for a given number of sensors (N) while satisfying the two-fold redundancy criteria [7]. However, RMRAs did not have CFEs and had to be found through exhaustive search. Although a few RMRA configurations ($6 \leq N \leq 10$) were found using integer programming, the computational burden of finding larger arrays was prohibitive. As a result, RMRA configurations for $N > 10$ are currently unknown. The symmetric nested array (symNA) has CFEs, but can be defined only for even values of N (starting from $N = 16$) [7]. The composite Singer array has a complicated formulation and exists only for certain values of N [10]. Similarly, the fractal sparse array could be formulated only for specific values of N [11].

Nevertheless, the array design formulated by Zhu *et al.* had CFEs valid for all array sizes starting from $N = 6$ [12]. For this, double difference bases (DDBs) available in number theory were employed as two-fold redundant arrays (2FRAs) [8]. However, it was later found that not every DDB can be a 2FRA [13]. Though every 2FRA is essentially a DDB, the converse is not necessarily true. In short, certain 2FRAs contain hidden dependencies that can disrupt the continuity of the difference coarray (DCA), a consequence that is against the notion of providing two-fold redundancy. Overall, it can be concluded that currently there are no TFRSA designs that (i) have CFEs, (ii) are defined for all array sizes from $N = 6$, and (iii) are free of any hidden dependencies. This study addresses these gaps by formulating a novel TFRSA that overcomes all the above limitations.

B. Need for robust sparse arrays in future systems

Recent literature and ongoing 6G standardization efforts suggest the transformative and prominent role of sparse arrays in next-generation communication technologies, such as integrated sensing and communications (ISAC) and extra-large multiple-input multiple-output (XL-MIMO) systems [14], [15], [16]. In this context, it is beneficial to have robust sparse array designs handy, as they provide seamless and accurate direction of arrival (DOA) estimation during sensor failures—unlike maximally economic sparse arrays (MESAs), which cannot tolerate the failure of even a single sensor without affecting their DCAs and DOA estimation capabilities. To work around the coarray discontinuities in MESAs, sophisticated signal processing methods such as array diagnosis and coarray interpolation must be employed. However, these methods are computationally expensive and unsuitable for real-time implementations [17], [18], [19]. All these factors favor the potential use of multi-fold redundant sparse arrays in future systems. Moreover, resilient sparse arrays are highly desirable across a range of applications, including audio source localization, radar, sonar, and biomedical systems. Following are the unique contributions of this work: -

- A new TFRSA formulation that meets all the desired criteria is proposed. It provides CFEs for sensor positions and can be realized for sensor counts greater than or equal to six.
- It is shown that not every DDB can be a TFRSA. A simple test that differentiates DDBs from actual TFRSAs is proposed and a MATLAB code is provided for the same. The term '*Hidden Essential Sensor (HES)*' is coined for the first time.
- It is also shown that the proposed array exhibits a satisfactory radiation pattern, free from the sidelobe curse typically associated with sparse arrays. Moreover, it provides $O(N)$ degrees of freedom (DOFs), as required by modern wireless technologies.

The remainder of this paper is organized as follows. Section II discusses the relevant sparse array terminology and constraints to be followed for designing TFRSAs. Section III explains the mathematical framework for obtaining single-fold and two-fold redundant arrays from a ULA by repositioning one or two sensors, respectively. Section IV describes a test to verify the robustness of TFRSAs and highlights the shortcomings of the existing approaches. Section V presents a modified formulation for obtaining TFRSAs along with relevant mathematical proofs and numerical examples. Section VI compares the radiation characteristics of the proposed array with those of a ULA. Section VII outlines the limitations and discusses the future scope. Finally, section VIII concludes the paper.

II. Theoretical Background

A. Sparse Array Terminology

The important terms related to sparse arrays and coarray domain processing are described below.

- *Physical Array* $\mathbb{S}$ – Sensor positions in the physical array, normalized to half the wavelength, that is, $\lambda/2$.
- *Aperture* L – Spacing between the first and last sensors in the physical array $\mathbb{S}$.
- *Difference Set* $\mathbb{Z}$ – Set of all possible self- and cross-differences between the sensor positions $\mathbb{S}$.
- *Spatial Lag* – Each entry in the difference set $\mathbb{Z}$.
- *Difference Coarray (DCA)* $\mathbb{D}$ – Sorted and non-repeating entries extracted from $\mathbb{Z}$
- *Holes* – Missing spatial lags in $\mathbb{Z}$ and/or $\mathbb{D}$
- *Weight* – The number of times a given spatial lag appears in the difference set $\mathbb{Z}$.
- *Degrees of Freedom (DOFs): –* Refer to the number of entries in DCA ($\mathbb{D}$). An array with aperture L and a hole-free DCA enjoys $2L+1$ DOFs, as the DCA contains all spatial lags from $[-L,\ L]$. DOFs quantify the number of targets or source angles that the array can resolve during DOA estimation.
- *Weight Function* – List of weights corresponding to each spatial lag in the array.
- *Essential sensors* – A sensor whose removal from the array alters or shrinks the DCA. For a TFRSA, sensors located at 0 and L are considered essential.
- *Hidden Essential Sensor (HES)* – An essential sensor at any position other than 0 and L. An appropriately designed TFRSA should not consist of HESs.
- *Fragility* – The 1-fragility of a sparse array is defined as the ratio of problematic single sensor failures divided by the total number of single sensor failures. In summary, the number of essential sensors divided by the total number of sensors in the array.
- *Signal Model* – The signal model is based on second order difference coarray processing using the coarray MUSIC algorithm. Because this signal model is widely available in the existing literature [3], [20], [21], [22], it is omitted from the discussion to save space.

B. TFRSA design rules

Ideally, a TFRSA should have two essential sensors to achieve a fragility of $2/N$. However, some existing TFRSA formulations contain three essential sensors or exhibit a fragility of $3/N$. There is no guarantee that such arrays operate consistently under all instances of single sensor failures. A correct TFRSA formulation abides by the design rules explained below.

1) Healthy Scenario

All spatial lags from $[-(L-1),(L-1)]$ shall have a weight of two or more i.e., $w(i)\geq 2;\ 0\leq i\leq L-1$. The spatial lag L occurs only once i.e., $w(L)=1$. The negative weights automatically follow, owing to the even symmetry $w(-m)=w(m)$.

2) Single Sensor Failure Scenario

Barring the failure of sensors at 0 and L, any single sensor should not introduce holes in the DCA. If a hole is formed in the DCA during any instance of single sensor failure, the failed sensor is classified as an HES.

III. Formulation of Two-Fold Redundant Sparse Array (TFRSA)

The following framework is proposed to formulate single-fold and two-fold sparse arrays inspired by ULAs.

A. ULA-inspired sparse array by relocating a single sensor

Upon analyzing the properties of a ULA through the lens of sparse arrays, we were intrigued by the possibility of constructing a sparse array that differs from a ULA by only a single sensor. To this end, the last sensor of an N-element ULA was repositioned to a grid point x.

The set $\mathbb{S}_1 = [0, 1, 2, 3, \ldots, N-2 \quad x]$ represents the configuration of the proposed ULA-inspired single-fold array. The ULA segment from 0 to $N-2$ ensures that all the spatial lags from 0 to $N-2$ occur at least once. The position x shall be chosen such that the DCA of $\mathbb{S}_1$ is hole-free from 0 to x. This is possible when the difference between x and the last sensor in the ULA segment $(N-2)$ generates the next required lag i.e., $N-1$. Hence, we have

$$x - (N-2) = N - 1$$

$$x = 2N - 3 \quad (1)$$

Hence, the complete sparse array is given by

$$\mathbb{S}_1 = [0, 1, 2, 3, \ldots, N-2 \quad 2N-3] \quad (2)$$

Therefore, a holefree SLA with almost double the aperture of the ULA is obtained by repositioning a single sensor from the ULA. The strategic placement of the sensor at $x = 2N-3$ ensures a hole-free DCA from 0 to $2N-3$. The proof is as follows. Lags 0 to $N-2$ occur at least once because of the ULA segment from 0 to $N-2$. The remaining lags from $N-1$ to $2N-3$ are generated owing to the relative spacing between the last sensor and each of the sensors in the ULA segment.

B. ULA-inspired TFRSA

We extend the framework described above to formulate a TFRSA. To this end, we selected two sensors from the far end of the ULA and placed them at positions $y-1$ and y, respectively. The tentative array is given by $\mathbb{S}_2 = [0, 1, 2, \ldots, N-3, \quad y-1, y]$. The ULA segment from 0 to $N-3$ ensures that all the lags from 0 to $N-4$ occur at least twice. Lag $N-3$ occurs only once within the ULA segment. The corner sensor at y generates all unrepresented lags from $N-2$ to itself. Placing another sensor immediately adjacent to this (at $y-1$) helps generate nearly the same lags once again, except for the final lag y. Instead of the final lag y, the sensor at $y-1$ contributes to repeating the spatial lag $N-3$. Thus, placing two sensors next to each other ensures that each spatial lag from 0 to $y-1$ has a multiplicity of at least two, as desired.

The spacing between y and the last sensor in the ULA segment should generate the next required lag, $N-2$. Hence,

$$y - (N-3) = N - 2$$

$$y = 2N - 5 \quad (3)$$

which leads to the following array configuration

$$\mathbb{S}_2 = [0, 1, 2, \ldots, N-3 \quad 2N-6 \quad 2N-5] \quad (4)$$

The notation $\mathbb{S}_i$ not only signifies the number of off-ULA sensors but also that the array offers i-fold redundancy. To gain an intuitive understanding of the respective array geometries, we evaluated arrays $\mathbb{S}_1$ and $\mathbb{S}_2$ for $N = 10$ and compared their sensor positions with those of a 10-element ULA, as shown in Fig. 1.

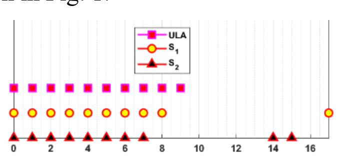


Fig. 1. Sensor Positions of ULA, $\mathbb{S}_1$, and $\mathbb{S}_2$ for $N = 10$

The array $\mathbb{S}_2$ represents a double difference base (DDB) because it can generate all spatial lags between 0 to $2N-6$ at least twice. It satisfies the weight function requirements of a TFRSA in the healthy case. This can be verified through a numerical example. Consider $N = 10$. The array contains sensors at [0, 1, 2, 3, 4, 5, 6, 7, 14, 15], leading to the weight function shown in Fig. 2. As seen, all spatial lags from 0 to 14 have a weight of two or more, as expected. The next step is to check whether this array can provide a hole-free DCA in all permitted instances of single sensor failures. To this end, a systematic test is presented in the next section.

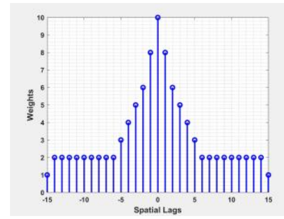


Fig. 2. Weight function of the 10-element array $\mathbb{S}_2$

IV. Test For Robustness

Initially, we performed manual testing that involved removing sensors one at a time from the array and observing the resulting weight function in each case. However, we found this approach to be both tedious and error-prone. To address these issues, we developed a MATLAB program that can automatically simulate all permissible single sensor failures and assess the weight function/DCA continuity of the array. If the program encounters a DCA with holes during any instance of a single sensor failure, it increments the count variable (initially set to zero) and flags the failed sensor in that instance as a hidden essential sensor (HES).

After evaluating the array under all single sensor failures, we check whether the count value is still zero. If so, the array is declared two-fold redundant. Otherwise, it is considered fragile and the position of the HES that causes this vulnerability is displayed. The MATLAB code is available at https://github.com/profashish/two-fold-robustness-check.

A. Thorough failure analysis of the $\mathbb{S}_2$ array

The following representative samples of array $\mathbb{S}_2$ were tested using the procedure/program just discussed. The robustness check results are presented in Table I. It can be observed that the array $\mathbb{S}_2$ suffers from a HES at position $N-3$, irrespective of the sensor count (N) in the array.

Table I.
Robustness Checking for $\mathbb{S}_2$ Arrays of Various Sizes

Array Size *N*	**Sensor Positions**	**HESs (if any)**
5	[0, 1, 2, 4, 5]	2
6	[0, 1, 2, 3, 6, 7]	3
7	[0, 1, 2, 3, 4, 8, 9]	4
8	[0, 1, 2, 3, 4, 5, 10, 11]	5
9	[0, 1, 2, 3, 4, 5, 6, 12, 13]	6
10	[0, 1, 2, 3, 4, 5, 6, 7, 14, 15]	7
15	[0, 1, 2, 3,..., 12, 24, 25]	12
20	[0, 1, 2, 3, ..., 17, 34, 35]	17

B. Problem with HES a.k.a. $3/N$ fragility

Although arrays with $2/N$ fragility are truly robust against all instances of permissible single sensor failures, those with $3/N$ fragility (consisting of three essential sensors or one HES) are not. The robustness of the latter is questionable and depends on the position of the failed sensor. If the failed sensor happens to be the HES, the array's DOA estimation performance would be compromised owing to discontinuities in the DCA. In certain cases (when the number of source angles to be resolved is comparable to the span of the central hole-free portion of the DCA), the root mean square error (RMSE) between the true and estimated angles, can vary by almost two orders on the semi log scale (10^{-2} to 10^{0}), compared to the healthy case. Surprisingly, the same array provides seamless operation and accurate DOA estimation if the failed sensor is not an HES. This unpredictability can lead to undesired outcomes in critical tasks such as aerospace, medical imaging, or defense systems.

Hence, it is important to ensure that the TFRSAs maintain a consistent fragility of $2/N$ across all configurations. To this end, the array formulation in Section III must be modified in such a way that the updated array is free from HESs.

V. Correct TFRSA Formulation (Proposed Array)

The HES at position $N-3$ in the array $\mathbb{S}_2$, defined in (4), could have been avoided if the last sensor had been placed at $2N-6$ instead of $2N-5$.

A. *Characteristics of the proposed array*

As per the updated configuration, the proposed array has its first sensor at zero and the last sensor at $2N-6$. Therefore, the complete array is given by

$$\mathbb{S}_{2,new} = [0, 1, 2, \ldots, N-3 \quad 2N-7 \quad 2N-6] \qquad (5)$$

This new (proposed) array provides an aperture of $L = 2N-6$ and $D = 4N-11$ DOFs for N sensors. Although it reduces the array aperture by one unit compared to array $\mathbb{S}_2$, it keeps the problem of HESs at bay. For this array to be sparse, the aperture $2N-6$ should be greater than $N-1$ (the ULA aperture). The smallest value of N for which the inequality $2N-6 > N-1$ holds is $N = 6$. Therefore, the array $\mathbb{S}_{2,new}$ can be defined for all $N \geq 6$.

B. *Proof for DCA continuity*

It must be proved that the proposed array satisfies the properties listed in Section II-B.

1) *Healthy case*

- The ULA segment from 0 to $N-3$ guarantees that all the differences from 0 to $N-4$ occur at least twice. The difference $N-3$ occurs only once in the ULA segment.
- The sensor at $2N-7$ creates all the differences from $N-4$ to itself ($2N-7-\{0{:}1{:}N-3\}$). This ensures that lags $N-2$ to $2N-7$ are generated once. The sensor at $2N-7$ adds one more instance to the spatial lag $N-3$, making its weight as two.
- The sensor at $2N-6$ ensures that all differences from $N-3$ to itself are obtained ($2N-6-\{0{:}1{:}N-3\}$). All lags from $N-2$ to $2N-7$ appear for the second time, proving that the proposed configuration can generate all lags from 0 to $2N-7$ at least twice. As expected, the final lag $2N-6$ is generated only once.

2) *Single sensor failure case*

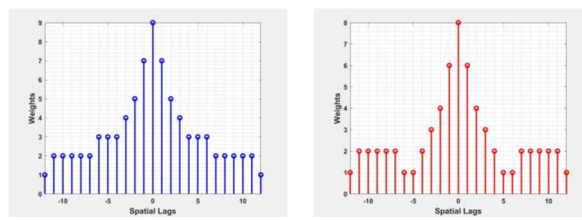
The proof for this is through reasoning. Because the sensors at 0 and L are essential, they are always assumed to be functional. This ensures that spatial lag L is never missed. Because all remaining spatial lags have a weight of two or more, assuming a hypothetical sensor $i \in [1, \ L-1]$ whose failure affects one instance of all lags from 0 to $L-1$, there would still be at least one more instance left that can generate all these lags. Thus, all lags from 0 to L occur at least once even during a single sensor failure, leading to a hole-free DCA.

C. *Numerical Examples*

Representative samples of the proposed array, namely, $\mathbb{S}_{2,new}$ are listed in Table II. Failure analysis using the program discussed in Subsection IV-A showed that these arrays are devoid of HESs. Although we tested many arrays, only a few could be listed here. Fig. 3 shows the weight function graph of the proposed array $\mathbb{S}_{2,new}$ for $N = 9$ under both healthy and faulty conditions. The faulty case corresponds to the failure of the sensor at $N-3 = \{6\}$ which was essential for the array $\mathbb{S}_2$. However, as shown in Fig. 3b, the failure of this sensor does not create holes in the DCA of $\mathbb{S}_{2,new}$, as none of the weights are zero. A similar procedure was repeated for sensor faults at other positions, only to confirm that the array $\mathbb{S}_{2,new}$ indeed has no HESs.

Table II.
Corrected TFRSA Configurations: $\mathbb{S}_{2,new}$

N	**Sensor Positions**	**HES**
6	[0, 1, 2, 3, 5, 6]	Nil
7	[0, 1, 2, 3, 4, 7, 8]	Nil
8	[0, 1, 2, 3, 4, 5, 9, 10]	Nil
9	[0, 1, 2, 3, 4, 5, 6, 11, 12]	Nil
10	[0, 1, 2, 3, 4, 5, 6, 7, 13, 14]	Nil
20	[0, 1, 2, 3, …, 17, 33, 34]	Nil



(a) (b)

Fig. 3. Weight function with and without the sensor at {6} for $N = 9$

Overall, the array $\mathbb{S}_{2,new}$ satisfies almost all the desirable properties from a sensing perspective: (i) CFEs for sensor positions, (ii) absence of HESs, (iii) a consistent fragility of $2/N$, and (iv) a formulation that is valid for all array sizes starting from $N = 6$. Moreover, it aligns with the recent trend of designing SLAs with $O(N)$ DOFs [16], [22].

VI. Radiation Pattern/Array Factor

Besides the aforementioned advantages, the proposed array $\mathbb{S}_{2,new}$ exhibits an attractive radiation pattern. Unlike other sparse arrays, it does not suffer from high sidelobe levels and/or grating lobes. As shown in Fig. 4, its radiation pattern broadly resembles that of a N-element ULA. As can be seen, there are no spurious sidelobes. The pointed edge at the top ensures that array $\mathbb{S}_{2,new}$ has a lower half-power beam width (HPBW) than the ULA, as expected, given its larger aperture.

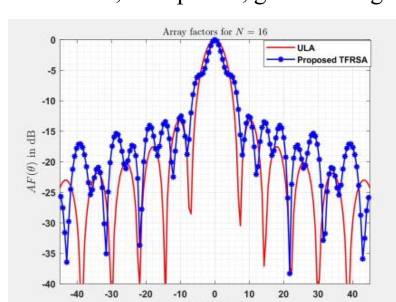


Fig. 4. Radiation patterns of ULA and the proposed array for $N = 16$.

To complete the comparison, we also plotted the radiation patterns of the Dong's 2FRA [9] and the symmetric nested array (symNA) [7] for $N = 16$, as shown in Fig. 5. As common to sparse arrays [1], [21], the first sidelobes of both the arrays were found to be just -6 to -7 dB below the main lobe. The respective configurations of the 2FRA and the symNA for $N = 16$ are [0, 1, 8, 9, 18, 19, 28, 29, 38, 39, 40, 41, 42, 43, 44, 45] and [0, 1, 2, 3, 4, 5, 9, 10, 14, 15, 19, 20, 21, 22, 23, 24].

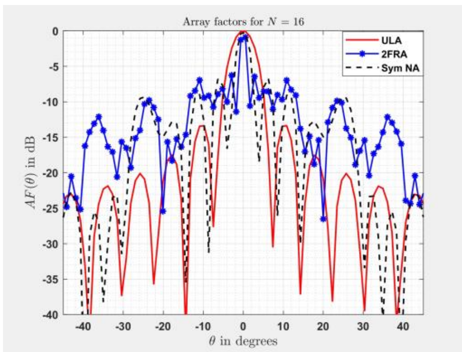


Fig. 5. Radiation patterns of ULA, 2FRA, and SymNA for $N = 16$

VII. Limitations and Future Scope

Although there are no major limitations in the proposed design, the large ULA segment (consisting of $N - 2$ sensors) at the beginning of the array results in higher primary weights which makes the array highly susceptible to mutual coupling.

In the future, this framework can be extended to find three and four-fold redundant sparse linear arrays. It can also be used to propose novel planar sparse arrays with two-fold redundancies.

VIII. Conclusion

This study solves a longstanding challenge in the area of sparse array design, namely, the formulation of two-fold redundant sparse arrays (TFRSAs) that can be characterized using closed-form expressions (CFEs) valid for all array sizes. The proposed approach overcomes almost all design flaws in existing formulations. For instance, unlike RMRAs, the proposed array has CFEs for the sensor positions. Unlike the symmetric nested array and composite singer array, the proposed array is valid for all sensor counts beyond $N = 5$. Unlike the 2FRAs, it does not consist of HESs. Apart from this, a systematic robustness test to detect the presence of HESs has also been proposed to differentiate TFRSAs from DDBs. A MATLAB program has also been provided for the same. Overall, the work presented here could be a new beginning in the study of multi-fold redundant sparse arrays.

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