Relations Between Analytic Spectral and Singular Value Decompositions

Stephan Weiss[1] and Giovanni Barbarino[2]

[1]Department of Electronic & Electrical Engineering, University of Strathclyde, Glasgow G1 1XW, Scotland

[2]Mathematics and Operational Research Unit, University of Mons, Belgium

stephan.weiss@strath.ac.uk, giovanni.barbarino@umons.ac.be

***Abstract*—We compare the existence and uniqueness of the analytic singular value decomposition (SVD) of a matrix $\boldsymbol{A}(z)$ to that of the analytic spectral or eigenvalue decomposition (EVD) of $\boldsymbol{R}(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$. Both cases require oversampling if the matrices are connected to multiplexing operations. Additionally, the analytic SVD may require additional oversampling due to zero crossings of its singular values, which, different from ordinary matrices, cannot necessarily be constrained to be non-negative. It has recently been shown that oversampling can be compensated in parts by permitting singular values to be complex-valued holomorphic on the unit circle, and additionally for the SVD to perform a block-diagonalisation to pseudo-circulant subblocks. We demonstrate here that complex-valued singular values can also be motivated through fractional delay factors, link the SVD-variants to the analytic spectral decomposition, and discuss some aspects of the uniqueness of both the SVD and the spectral decomposition.**

I. Introduction

Linear algebraic decompositions are an important tool in many disciplines, with the singular value decomposition (SVD) [1], [2] and the spectral decomposition [3] amongst the important 'big six' linear algebraic operations [4]. The singular value decomposition plays an important role generally in signal processing [5]–[7], with important applications for example for the factorisation of multiple-input multiple-output communications channels [8]. Similarly, the spectral decomposition — an eigenvalue decomposition (EVD) applied to a Hermitian matrix — is central to many statistical and array processing problems and optimisations [9]. When considering broadband multichannel problems, advantageous descriptions use matrices of polynomials or generally of analytic functions. To solve such problems, standard linear algebraic factorisations can only e.g. diagonalise such a matrix for a specific parameter value. A complete diagonalisation requires polynomial or analytic matrix factorisations [9], [10].

The existence of an analytic EVD for parahermitian matrices — i.e. a spectral decomposition — has been addressed in [11]–[13], and that of an SVD in [13], [14]. In some cases, these decompositions only exist with analytic factors if the matrix to be decomposed is oversampled, whereby the oversampling factor can vary across the SVD and EVD. One reason for oversampling both SVD and EVD lies in any multiplexing that is embedded in the matrices to be decomposed. For the SVD, where analytic singular values may possess zero crossings, an odd number of such transitions will require an additional oversampling by two.

In order to curb or even eliminate the need for oversampling, for the analytic SVD a number of alternative factorisations have been proposed in [15], whereby it may suffice to tolerate complex-valued singular values holomorphic on the unit circle, or even a block-diagonal form for the decomposition. In this paper, the aim is to firstly harmonise these results for a matrix $\boldsymbol{A}(z) : \mathbb{C} \rightarrow \mathbb{C}^{M\times N}$ with the analytic spectral factorisation of $\boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$, where $\boldsymbol{A}^{\mathrm{P}}(z) = \{\boldsymbol{A}(1/z^*)\}^{\mathrm{H}}$ is the parahermitian transpose of $\boldsymbol{A}(z)$. Secondly, we aim to establish aspects of the non-uniqueness for these proposed factorisations, which is important when seeking algorithmic implementations.

In the following, Sec. II will introduce a unified model for $\boldsymbol{A}(z)$ and a parahermitian matrix $\boldsymbol{R}(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$. Thereafter, Sec. III will recall the analytic SVD in [13], [14], and its variations in [15]. Sec. IV will address equivalent formulations for the analytic spectral decomposition. A summary and consequences for algorithm development are presented in Sec. V.

II. Signal / Source Model

A. Innovation Filters

In order to describe the 2nd order statistics of a measurement vector $\mathbf{x}[n] \in \mathbb{C}^M$ with complex Gaussian elements that exhibit both temporal and spatial correlation, we use an innovation filter [16] to tie $\mathbf{x}[n]$ to a vector of N temporally and spatially uncorrelated Gaussian signals in $\mathbf{u}[n] \in \mathbb{C}^N$ with zero mean and unit variance. The innovation filter here is a system matrix $\mathbf{A}[n]$ as show in Fig. 1, whose element in the mth row and ℓth column represents the impulse response between the ℓth excitation in $\mathbf{u}[n]$ and the mth measurement in $\mathbf{x}[n]$, such that overall with $\mathbf{A}[n] \in \mathbb{C}^{M\times N}$ we have

$$\mathbf{x}[n] = \sum_{\nu} \mathbf{A}[\nu]\mathbf{u}[n-\nu] \; . \tag{1}$$

In the z-domain, $\boldsymbol{A}(z) = \sum_n \mathbf{A}[n]z^{-n}$ or for short $\boldsymbol{A}(z) \;\bullet\!\!-\!\!\circ\; \mathbf{A}[n]$, is a matrix of transfer functions, which we assume to be analytic in $z \in \mathbb{C}$.

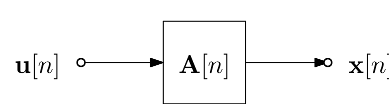

Fig. 1. Innovation filter model for the generation of the measurement $\mathbf{x}[n]$.

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B. Space-Time Covariance

The formulation of the space-time covariance matrix [9], [17],

$$\mathbf{R}[\tau] = \mathcal{E}\big\{\mathbf{x}[n]\mathbf{x}^{\mathrm{H}}[n-\tau]\big\} \ , \qquad (2)$$

can exploit the source model of (1), where $\mathcal{E}\{\cdot\}$ is the expectation operator. Thus, for the z-transform of the space-time covariance, the cross-spectral density (CSD) matrix $\boldsymbol{R}(z) \bullet\!\!-\!\!\circ \mathbf{R}[\tau]$, we have [18]

$$\boldsymbol{R}(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z) \ , \qquad (3)$$

with the parahermitian transpose $\boldsymbol{A}^{\mathrm{P}}(z) = \{\boldsymbol{A}(1/z^*)\}^{\mathrm{H}}$ [17]. Therefore, the space-time covariance and CSD matrices can be directly related to the innovation filter $\boldsymbol{A}(z)$.

C. Multiplexed System

Of particular importance to the existence of the analytic SVD and EVD are innovation models that relate to block-filtering [19], [20], where multiple inputs may be multiplexed across a smaller number of channels at a higher rate, and the outputs arise from demultiplexing. A simple example with a single input single output system $H(z)$ and multiplexing by $F = 2$ is shown in Fig. 2. For the general case $F \in \mathbb{N}$, $\boldsymbol{A}(z)$ will be a pseudo-circulant matrix, such that

$$\boldsymbol{A}(z) = \begin{bmatrix} H_0(z) & H_1(z) & \ldots & H_{F-1}(z) \\ z^{-1}H_{F-1}(z) & H_0(z) & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ z^{-1}H_1(z) & \ldots & z^{-1}H_{F-1}(z) & H_0(z) \end{bmatrix}, \qquad (4)$$

where entries are formed by the F polyphase components $H_f(z)$, $f = 0, \ldots, (F-1)$, of $H(z) = \sum_{f=0}^{F-1} z^{-f} H_f(z^F)$ [19].

Any pseudo-circulant matrix can be diagonalised by $\boldsymbol{W}_F(z) = \mathrm{diag}\big\{1, z^{-1}, \ldots, z^{-F+1}\big\}\,\mathbf{T}_F$, where $\mathbf{T}_F$ is an F-point discrete Fourier transform matrix normalised to be unitary, such that [17], [20]

$$\boldsymbol{A}(z) = \boldsymbol{W}_F(z^{1/F})\boldsymbol{D}(z^{1/F})\mathbf{W}_F^{\mathrm{P}}(z^{1/F}) \ , \qquad (5)$$

with

$$\boldsymbol{D}(z) = \mathrm{diag}\Big\{H(z), H(z\mathrm{e}^{\mathrm{j}\frac{2\pi}{F}}), \ldots, H(z\mathrm{e}^{\mathrm{j}\frac{2\pi(F-1)}{F}})\Big\} \ . \qquad (6)$$

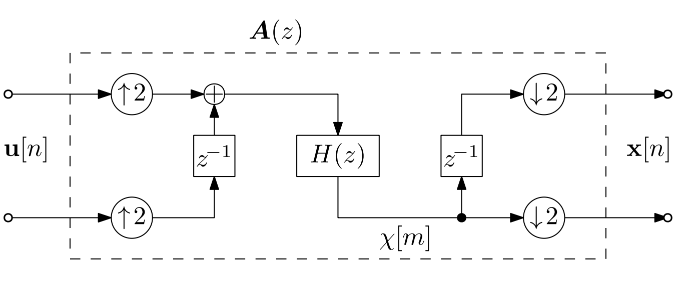

Fig. 2. Innovation filter $\boldsymbol{A}(z)$ arising through multiplexing by $F = 2$ across a system $H(z)$.

The factors on the r.h.s. of (5) only become analytic if $\boldsymbol{A}(z)$ is oversampled F-times. Since (5) is necessary and sufficient for a pseudo-circulant matrix [14], [17], a product of pseudo-circulant matrices remains pseudo-circulant. Specifically for $\boldsymbol{R}(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$ we have

$$\boldsymbol{R}(z) = \boldsymbol{W}_F(z^{1/F})\boldsymbol{D}(z^{1/F})\boldsymbol{D}^{\mathrm{P}}(z^{1/F})\mathbf{W}_F^{\mathrm{P}}(z^{1/F}) \ , \qquad (7)$$

where the term $\boldsymbol{D}(z^{1/F})\boldsymbol{D}^{\mathrm{P}}(z^{1/F})$ is diagonal and parahermitian i.e. real-valued on the unit circle. However, any fractional powers of z relate to Puiseux series [13], and for $F > 1$ are not analytic in $z \in \mathbb{C}$.

III. Analytic Singular Value Decomposition

We next consider an analytic SVD of $\boldsymbol{A}(z)$. There are different choices. A direct extension of the SVD of an ordinary matrix [3], as well as of the analytic SVD in a real interval [21], [22] leads to singular values that are constrained to be real. Alternative forms, where the SVD yields complex valued or even block-diagonal factors via paraunitary operations, has been explored in [15]. We relate these forms to the source model of Sec. II.

A. Real-Valued Singular Values

For the analytic SVD we have [13], [14],

$$\boldsymbol{A}(z) = \boldsymbol{U}_1(z^{1/(\kappa F)})\boldsymbol{\Sigma}(z^{1/(\kappa F)})\boldsymbol{V}_1^{\mathrm{P}}(z^{1/(\kappa F)}) \ , \qquad (8)$$

where $\boldsymbol{U}(z)$ and $\boldsymbol{V}(z)$ are paraunitary, and $\boldsymbol{\Sigma}(z)$ is parahermitian and diagonal. The parahermitian property of $\boldsymbol{\Sigma}(z)$ implies that on the unit circle, $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ is real-valued. In order to make the r.h.s. in (8) analytic, it may be necessary to oversample $\boldsymbol{A}(z)$ by a factor κF. The factor F is required if $\boldsymbol{A}(z)$ is F-times multiplexed. The factor $\kappa \in \{1, 2\}$ relates to the singular values, and will be motivated below. For the remainder of this paper, we assume that all singular values, except those identical to zero, are distinct.

As a peculiar difference to the standard SVD for ordinary matrices, $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$ cannot be constrained to be non-negative in order to admit an analytic $\boldsymbol{\Sigma}(z)$; this has been known for the analytic SVD on a real interval [21], [22], and has also recently been established for the factorisation in the discrete time case [13], [14]. In order to induce a 2π-periodicity of $\boldsymbol{\Sigma}(\mathrm{e}^{\mathrm{j}\Omega})$, every eigenvalue must possess an even number of zero crossings on the unit circle, which may necessitate $\kappa = 2$. We explore this by way of an example.

Example 1: Consider the matrix

$$\boldsymbol{A}(z) = \frac{1}{2}\begin{bmatrix} 1 + z^{-1} + 2z^{-2} & 1 - z^{-1} \\ 1 - z^{-1} & 1 + z^{-1} + 2z^{-2} \end{bmatrix} \ , \qquad (9)$$

which is not related to multiplexing and hence cannot be brought into a pseudo-circulant form. It possesses an analytic SVD

$$\boldsymbol{A}(z) = \underbrace{\mathbf{T}_2}_{\boldsymbol{U}_1(z^{\frac{1}{2}})} \underbrace{\begin{bmatrix} z + z^{-1} & \\ & z^{\frac{1}{2}} + z^{-\frac{1}{2}} \end{bmatrix}}_{\boldsymbol{\Sigma}(z^{\frac{1}{2}})} \underbrace{\mathrm{diag}\Big\{z^{-1}, z^{-\frac{3}{2}}\Big\}\,\mathbf{T}_2}_{\boldsymbol{V}_1^{\mathrm{P}}(z^{\frac{1}{2}})} . \qquad (10)$$

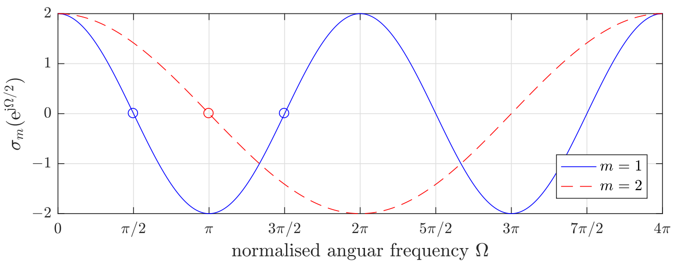

Fig. 3. Singular values of Example 1 with zero crossings on the interval $\Omega \in [0;\, 2\pi)$ marked by circles.

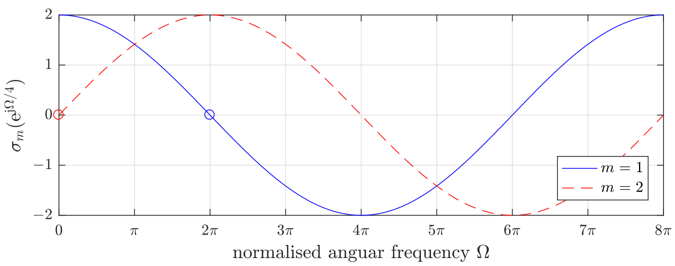

Fig. 4. Modulated singular values of Example 2 with zero crossings within the first 2π-cycle of Ω marked by circles.

The singular values $\sigma_1(z^{\frac{1}{2}}) = z+z^{-1}$ and $\sigma_2(z^{\frac{1}{2}}) = z^{\frac{1}{2}}+z^{-\frac{1}{2}}$ are depicted in Fig. 3. Because $\sigma_2(e^{j\Omega/2})$ only has a single zero crossing on the interval $0 \leq \Omega < 2\pi$, it is only 4π-periodic, and r.h.s. factors in (10) only become analytic if oversampled by $\kappa = 2$. Also, if singular values were constrained to be non-negative, i.e. if they were reflected at zero in Fig. 3, they would become non-differentiable in those reflection points and hence would not be analytic. △

Example 2: As a slightly modified example from [14], consider the matrix $\boldsymbol{A}(z) = [z^{-1},\ 1;\ 1,\ 1]$. With $\mathbf{J}$ the reverse identity, $\mathbf{J}\boldsymbol{A}(z)$ is pseudo-circulant and obtained from a system $H(z) = 1+z^{-1}$ via multiplexing by $F = 2$ as shown in Fig. 2. For an analytic SVD, it is possible to write

$$\boldsymbol{A}(z) = \underbrace{\mathbf{J}\boldsymbol{W}_2(z^{\frac{1}{2}})}_{\boldsymbol{U}_1(z^{\frac{1}{4}})} \underbrace{\begin{bmatrix} z^{\frac{1}{4}}+z^{-\frac{1}{4}} & \\ & z^{\frac{1}{4}}+z^{-\frac{1}{4}} \end{bmatrix}}_{\boldsymbol{\Sigma}(z^{\frac{1}{4}})} \underbrace{z^{-\frac{1}{4}}\boldsymbol{W}_2^{\mathrm{P}}(z^{\frac{1}{2}})}_{\boldsymbol{V}_1^{\mathrm{P}}(z^{\frac{1}{4}})} . \quad (11)$$

The singular values $\sigma_m(e^{j\Omega/4})$, $m = 1, 2$, are depicted in Fig. 4, which are modulated versions of each other akin to (6), and require oversampling with $F = 2$ due to multiplexing and $\kappa = 2$ due to each of the singular values only possessing one zero crossing within any $2\pi F$-interval of Ω. Due to the overall 8π-periodicity of $\sigma_m(e^{\Omega/4})$, (11) must be $\kappa F = 4$-fold oversampled in order to ensure that the r.h.s. is analytic. △

By constraining $\Sigma(e^{j\Omega/(\kappa F)})$ to be real-valued on the unit circle, apart from their ordering and a sign change, the singular values are therefore unique. If the singular values are distinct, then their associated left- and right-singular vectors are unique up to arbitrary allpass functions that is coupled across both vectors [14]: with $\boldsymbol{U}_1(z)$ and $\boldsymbol{V}_1(z)$ containing valid singular vectors, then, assuming $M = N$, so are $\boldsymbol{U}_1(z)\boldsymbol{\Phi}(z)$ and $\boldsymbol{V}_1(z)\boldsymbol{\Phi}(z)$, with $\boldsymbol{\Phi}(z) = \mathrm{diag}\{\varphi_1(z), \ldots, \varphi_M(z)\}$ a matrix of arbitrary analytic allpass filters.

B. Complex-Valued Singular Values

In [15], a complex-valued analytic SVD

$$\boldsymbol{A}(z) = \boldsymbol{U}_2(z^{1/F})\boldsymbol{S}(z^{1/F})\boldsymbol{V}_2^{\mathrm{P}}(z^{1/F}) \quad (12)$$

is shown to always exist, where $\boldsymbol{U}_2(z)$ and $\boldsymbol{V}_2(z)$ are paraunitary. The matrix $\boldsymbol{S}(z)$ is diagonal but $\boldsymbol{S}(e^{j\Omega})$ may be complex-valued holomorphic; oversampling w.r.t. $\kappa = 2$ is no longer required. As an example, the diagonalisation of a pseudo-circulant system via (5) and (6) demonstrates this without placing any restrictions on $H(z)$.

In order to relate $\boldsymbol{S}(z)$ in (12) to $\boldsymbol{\Sigma}(z)$ in (8), it suffices to consider individual singular values, say $s(z)$ and $\sigma(z^{1/\kappa})$. We treat $s(z)$ as an $M = 1$ dimensional matrix which we subject to an analytic SVD of the type in (8). Note that since $M = 1$, there can be no multiplexing and we must have $F = 1$ and can focus on oversampling by κ alone. The analytic SVD on the unit circle yields

$$s(e^{j\Omega}) = \sigma'(\Omega)v'(\Omega) , \quad (13)$$

where $\sigma'(\Omega) \in \mathbb{R}$. Since $v'(\Omega)$ is an allpass filter with $|v'(\Omega)| = 1$, $s(e^{j\Omega})$ and $\sigma'(\Omega)$ share the same zero crossings. Since a bin-wise SVD must yield the same values for $\Omega = 0$ and $\Omega = 2\pi$, an analytic SVD must yield $\sigma'(0) = -\sigma'(2\pi)$ in the case that the number of zero crossings is odd. Then $\sigma'(\Omega)$ is 4π-periodic, and must be oversampled by $\kappa = 2$ in order to become analytic.

Alternatively, in the case of an odd number of zero crossings we can align the sign between $\sigma'(\Omega)$ and $\sigma'(\Omega-2\pi)$ by instead considering $\sigma'(\Omega)e^{-j\Omega/2}$. Note that $e^{-j\Omega/2}$ is the frequency response of a half-sample fractional delay $z^{-\frac{1}{2}}$; if required, both the singular value and its corresponding left- or right-singular vector can each share such a half-sample delay. Observe that in Example 1 with (10) and in Example 2 with (11), it is the singular values and their corresponding right-singular vectors that share fractional delays. It is therefore possible to write

$$\boldsymbol{A}(z) = \boldsymbol{U}_2(z^{1/F})\boldsymbol{\Sigma}(z^{1/(\kappa F)})\boldsymbol{V}_1^{\mathrm{P}}(z^{1/(\kappa F)}) . \quad (14)$$

This has been recognised in [15], even though without the system-theoretic reference to fractional delays.

Example 3: Recalling $\boldsymbol{A}(z)$ from Example 1, we can instead write for its analytic SVD

$$\boldsymbol{A}(z) = \underbrace{\mathbf{T}_2}_{\boldsymbol{U}_2(z)} \underbrace{\begin{bmatrix} z+z^{-1} & \\ & 1+z^{-1} \end{bmatrix}}_{\boldsymbol{S}(z)} \underbrace{z^{-1}\mathbf{T}_2}_{\boldsymbol{V}_2^{\mathrm{P}}(z)} . \quad (15)$$

The fractional delay of the second singular value in (10) has been absorbed into its corresponding right-singular vector. With the singular values shown in Fig. 5, this SVD exists now without the need for oversampling but the second singular value is no longer parahermitian and hence no longer real-valued on the unit circle. △

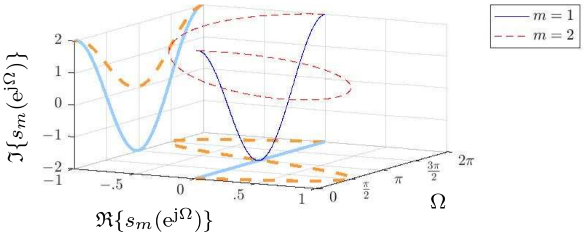

Fig. 5. Complex-valued 2π-periodic singular values of Example 3.

Without the real-valued constraint for $\boldsymbol{S}(z)$, $\boldsymbol{S}(z)\boldsymbol{\Psi}(z)$ also represents valid singular values if $\boldsymbol{\Psi}(z) = \mathrm{diag}\{\psi_1(z), \ldots, \psi_M(z)\}$ contains arbitrary allpass filters. Thus, w.r.t. Sec. III-A and with $M = N$, $\boldsymbol{U}_2(z)\boldsymbol{\Phi}(z)$ and $\boldsymbol{V}_2(z)\boldsymbol{\Phi}(z)\boldsymbol{\Psi}(z)$ also contain valid left- and right-singular vectors, and their ambiguities are now decoupled.

C. Pseudo-Circulant Block-Diagonalisation

As a further type of analytic singular value decomposition, arguments in [15] based on [14] have proven that it is possible to write, dispensing of any potential oversampling,

$$\boldsymbol{A}(z) = \boldsymbol{U}_3(z)\boldsymbol{C}(z)\boldsymbol{V}_3^{\mathrm{P}}(z) \ , \tag{16}$$

where $\boldsymbol{U}_3(z)$ and $\boldsymbol{V}_3(z)$ are paraunitary, and $\boldsymbol{C}(z)$ is a block-diagonal matrix. Each subblock of $\boldsymbol{C}(z)$ is a pseudo-circulant matrix; their dimensions depend on the different multiplexing factors incurred in $\boldsymbol{A}(z)$, assuming that different from the single multiplexing factor in Fig. 2, various parallel or nested multiplexed systems are possible. Nonetheless, the existence of (16) implies that any arbitrary multiplexing structure can brought into a parallel arrangement of multiplexed systems.

Example 4: Returning to Example 2, we have already implicitly established how $\boldsymbol{A}(z)$ can be brought into pseudo-circulant form. Therefore,

$$\boldsymbol{A}(z) = \underbrace{\mathbf{J}}_{\boldsymbol{U}_3(z)} \underbrace{\begin{bmatrix} 1 & 1 \\ z^{-1} & 1 \end{bmatrix}}_{\boldsymbol{C}(z)} \underbrace{\mathbf{I}}_{\boldsymbol{V}_3^{\mathrm{P}}(z)} \ . \tag{17}$$

represents the analytic SVD in the sense of (16). △

To explore the uniqueness of the block-diagonal matrix $\boldsymbol{C}(z)$ in (16), it is possible to exchange terms between the singular values and the left- and right-singular matrices that are paraunitary and themselves pseudo-circulant, recalling that products between pseudo-circulant matrices remain pseudo-circulant. This includes a number of possibilities of pseudo-circulant matrices that are constructed from allpass functions, but the details are beyond the scope of this paper.

IV. Analytic Spectral Decomposition

Below, we utilise the analytic SVD definitions to construct equivalent forms for an analytic spectral decomposition.

A. Real-Valued Diagonal Form

If the measurement vector $\mathbf{x}[n]$ in Fig. 1 is related to F-fold multiplexing via the source model $\boldsymbol{A}(z)$, then its CSD matrix admits the analytic EVD or spectral decomposition [11]–[13]

$$\boldsymbol{R}(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z) = \boldsymbol{Q}(z^{1/F})\boldsymbol{\Lambda}(z^{1/F})\boldsymbol{Q}^{\mathrm{P}}(z^{1/F}) \ . \tag{18}$$

The same result can be reached via the real-valued and complex-valued analytic SVDs. For the complex-valued analytic SVD, with (12) the direct equivalences w.r.t. (18),

$$\boldsymbol{Q}(z) = \boldsymbol{U}_2(z) \ , \tag{19}$$

$$\boldsymbol{\Lambda}(z) = \boldsymbol{S}(z)\boldsymbol{S}^{\mathrm{P}}(z) \ , \tag{20}$$

are straightforward: inspecting the diagonal elements of (20), on the unit circle, $\lambda_m(e^{j\Omega}) = s_m(e^{j\Omega})s_m^{\mathrm{H}}(e^{j\Omega}) = |s_m(e^{j\Omega})|^2$, $m = 1, \ldots, M$, the analytic eigenvalues are indeed non-negative real. For the relation to the real-valued analytic SVD, we exploit (14), and find

$$\boldsymbol{Q}(z) = \boldsymbol{U}_2(z) \tag{21}$$

$$\boldsymbol{\Lambda}(z) = \boldsymbol{\Sigma}(z^{1/\kappa})\boldsymbol{\Sigma}^{\mathrm{P}}(z^{1/\kappa}) \ . \tag{22}$$

On the r.h.s. of (22), the order of all spectral zeros is doubled and thus even, such that the product has no longer any zero crossings. Hence any fractional terms disappear even for $\kappa = 2$, i.e. again the analytic eigenvalues are non-negative real.

Example 5: For $\boldsymbol{A}(z)$ of Example 1 and its second singular value $\sigma_2(z^{1/2})$ in (10), with (22) we have

$$\lambda_2(z) = (z^{\frac{1}{2}} + z^{-\frac{1}{2}})(z^{-\frac{1}{2}} + z^{\frac{1}{2}}) = z + 2 + z^{-1} \ , \tag{23}$$

which indeed no longer contains fractional terms. Inserting the complex-valued analytic SVD in (15) for the same system matrix $\boldsymbol{A}(z)$ in Example 3, the use of (20), such that

$$\lambda_2(z) = (1 + z^{-1})(1 + z) = z + 2 + z^{-1} \ , \tag{24}$$

leads to the same result as (23). △

Thus, the analytic real- and complex valued SVDs are reconciled with the spectral decomposition. The additional allpass ambiguities of the complex-valued singular values over the real-valued ones are cancelled in the parahermitian product $\boldsymbol{S}(z)\boldsymbol{S}^{\mathrm{P}}(z)$, such that the analytic eigenvalues in $\boldsymbol{\Lambda}(z)$ are unique up to their ordering. The non-uniqueness of the eigenvectors in (18) is coupled to the allpass ambiguity of the left-singular vectors via (19) and (21), and agrees with the findings in [11], [12].

B. Pseudo-Circulant Block-Diagonalisation

Based on (16), the CSD matrix may be factorised as

$$\boldsymbol{R}(z) = \boldsymbol{Q}_3(z)\boldsymbol{L}(z)\boldsymbol{Q}_3^{\mathrm{P}}(z) \tag{25}$$

without any need for oversampling, with $\boldsymbol{Q}_3(z)$ a paraunitary matrix, and $\boldsymbol{L}(z)$ a parahermitian and pseudo-circulant block-diagonal matrix. W.r.t. (16), the factors are

$$\boldsymbol{Q}_3(z) = \boldsymbol{U}_3(z) \tag{26}$$

$$\boldsymbol{L}(z) = \boldsymbol{C}(z)\boldsymbol{C}^{\mathrm{P}}(z) \ , \tag{27}$$

whereby $\boldsymbol{L}(z)$ is both parahermitian and pseudo-circulant.

Example 6: For the CSD matrix $\boldsymbol{R}(z) = \boldsymbol{A}(z)\boldsymbol{A}^{\mathrm{P}}(z)$ derived from the analytic SVD in Example 4, we obtain

$$\boldsymbol{R}(z) = \underbrace{\mathbf{J}}_{\boldsymbol{Q}_3(z)} \underbrace{\begin{bmatrix} 2 & z+1 \\ 1+z^{-1} & 2 \end{bmatrix}}_{\boldsymbol{L}(z)} \underbrace{\mathbf{J}^{\mathrm{H}}}_{\boldsymbol{Q}_3^{\mathrm{P}}(z)} \ . \tag{28}$$

The matrix $\boldsymbol{L}(z)$ contains the $F = 2$ polyphase components $R_{\chi,f}(z)$ of the power spectral density $R_\chi(z) = \sum_{f=0}^{F-1} z^{-m} R_{\chi,f}(z^F)$ of $\chi[m]$ in Fig. 2, i.e. the signal being demultiplexed by F to form the measurement vector $\mathbf{x}[n]$. This power spectral density is given by $R_\chi(z) = H(z)H^{\mathrm{P}}(z) = z+2+z^{-1}$. Thus, we have $R_{\chi,0}(z) = 2$ and $R_{\chi,1}(z) = 1+z$, such that that the pseudo-circulant matrix

$$\boldsymbol{L}(z) = \left[\begin{array}{rr} R_{\chi,0}(z) & R_{\chi,1}(z) \\ z^{-1}R_{\chi,1}(z) & R_{\chi,0}(z) \end{array} \right] \quad (29)$$

agrees with (28). $\triangle$

The uniqueness of the decomposition in (25) has been previously unexplored. The matrix $\boldsymbol{Q}_3(z)$ matches the left-singular matrix in the pseudo-circulant SVD in (16), and hence shares at the least its non-uniqueness w.r.t. to arbitrary pseudo-circulant paraunitary matrices applied to every subblock of $\boldsymbol{C}(z)$. Such factors and their parahermitian transposes are also left- and right-multiplied to the block-pseudo-circulant $\boldsymbol{L}(z)$, but cancel since they are diagonalisable by the same paraunitary matrices $\boldsymbol{W}(z)$.

V. Discussion and Conclusions

In this paper, we have explored and compared the extension of two important factorisations, the singular value decomposition and the spectral decomposition, from ordinary matrices to matrices of analytic functions. This extension yields a number of curiosities, such as the need to admit singular values with sign changes if real-valuedness on the unit circle is to be maintained. Alternatively, we have followed the suggestion in [15] to define an SVD with two modifications: complex-valued singular values, as well as an SVD factorisation such that paraunitary operations yield a block-diagonal form with pseudo-circulant subblocks. The latter also ensures that an analytic SVD exists without the need for oversampling even in the case the SVD is applied to multiplexed systems. A similar factorisation including a pseudo-circulant block-diagonal form of analytic 'eigenvalues' rather than a complete diagonalisation has been suggested here for the analytic spectral or eigenvalue decomposition.

Additionally, we have commented on the uniqueness of the different decompositions, where some factorisations can be modified by allpass filters or pseudo-circulant systems relating to allpass functions. While this does not affect the analyticity of a solution, meaning that an arbitrarily close polynomial solution can be obtained by delays and truncations [10], it does impact on its order and therefore computational complexity. Such ambiguities are therefore important to understand when attempting to realise algorithmic solutions. While for the analytic diagonal spectral factorisation algorithms with proven convergence exist [23]–[25], solutions for the remaining decompositions are waiting to be explored. As an initial step, the complex value SVD with an algorithm based on [26] has been exploited to determine smooth precoders for MIMO multicarrier communications systems in [27].

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