Least-Squares Khatri-Rao Factorization of a Polynomial Matrix

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***Abstract*—The Khatri-Rao product is extensively used in array processing, tensor decomposition, and multi-way data analysis, where many applications rely on a least-squares (LS) Khatri-Rao factorization. In broadband sensor array problems, polynomial matrices effectively model frequency-dependent behaviors, necessitating extensions of conventional linear algebra techniques. This paper generalizes the LS Khatri-Rao factorization from ordinary to polynomial matrices by applying it to the discrete Fourier transform (DFT) sample points of polynomial matrices. Phase coherence across bin-wise Khatri-Rao factors is ensured via a phase-smoothing algorithm. The proposed phase smoothing method is validated through broadband angle-of-arrival (AoA) estimation for uniform planar arrays (UPAs), where the steering matrix is a polynomial matrix and can be represented as a Khatri-Rao product between decoupled steering matrices in azimuth and elevation directions.**

I. Introduction

Angle of arrival (AoA) estimation in narrowband array scenarios can be effectively addressed using conventional matrix algebra, especially the eigenvalue decomposition (EVD) [1]. Among the numerous harmonic retrieval algorithms, the spectral MUltiple SIgnal Classification (MUSIC) algorithm is a widely employed EVD-based technique [2, 3]. To circumvent the need for a grid-based search, its root-MUSIC variant [4] estimates the roots of a polynomial corresponding to the AoA of the sources impinging on the receiver antenna array.

The root-MUSIC method is particularly suitable for uniform linear arrays (ULAs), where the Vandermonde structure of the steering vector can be readily reformulated as a polynomial. In contrast for uniform planar arrays (UPAs), where the estimation of both elevation and azimuth angles is required, applying root-MUSIC directly is infeasible. The steering matrix for UPAs can be modelled as the Khatri-Rao product of two steering matrices, i.e., horizontal and vertical steering matrices [5, 6]. Consequently, a least squares (LS) Khatri-Rao factorization [7, 8] can be employed to separate the steering matrices, enabling the application of MUSIC or root-MUSIC independently for elevation and azimuth spatial angles without any need to pair angles as required in methods such as [9]. The Khatri-Rao factorization has also been utilized in various other signal processing applications, such as in channel and joint channel/symbol estimation for multi-antenna systems [7, 8, 10, 11], array processing [12], as well as in individual channel estimation for reconfigurable surfaces [13].

For broadband sensor arrays, the information on the AoA is embedded in the delay with which signals arrive at the different sensor elements. Therefore, lag information must be explicitly preserved, and quantities such as the space-time covariance matrix of the array are now functions of this lag parameter, thus turning into polynomial matrices when z-transformed [14]. Decompositions such as the polynomial EVD (PEVD) [15–18] can lead to generalisations of narrowband techniques to the broadband case. For ULAs, the MUSIC algorithm has been generalized to polynomial MUSIC (PMUSIC) [19], capable of performing both spatial and spectral estimation. Analogous to the narrowband case, the PMUSIC approach has been modified to polynomial root-MUSIC (PRMUSIC) [20], significantly lowering the computational complexity associated with PMUSIC by reducing the number expensive convolutions that need to be evaluated.

In this paper, we extend the Khatri-Rao factorisation from the narrowband case to that of broadband multiple-input multiple-output (MIMO) communication systems, and in particular to polynomial matrices which adequately model broadband MIMO systems. The proposed approach operates in the DFT domain, whereby an SVD is computed in each DFT bin. Due to a non-uniqueness of the SVD and in order to reestablish spectral coherence, phase smoothing must be applied to the left- and right-singular vectors. The DFT size is iteratively increased until a convergence metric is satisfied. As an example, the proposed algorithm is applied to an estimated broadband sensor array steering matrix of a UPA for broadband AoA estimation. While this paper does not focus on quantitatively comparing the computational cost reductions between PMUSIC and PRMUSIC with a LS Khatri-Rao factorisation for UPAs, our primary contribution lies in presenting the formulation and implementation of a LS Khatri-Rao factorisation for polynomial matrices.

The structure of this paper is as follows: Sec. II reviews an LS approximation of an ordinary matrix by a Khatri-Rao product. In Sec. III, this is extended to the case of polynomial matrices based on an analytic SVD with its algorithmic implementation outlined in Sec. IV. A worked example is presented in Sec. V, followed by an AoA estimation problem for a broadband UPA in Sec. VI. Conclusions are drawn in Sec. VII.

II. Best Least-Squares Approximation of a Matrix by a Khatri-Rao Product

Consider a matrix $\mathbf{A} \in \mathbb{C}^{MN\times P}$ which can be approximated as a Kathri-Rao product of two matrices $\mathbf{G} = [\mathbf{g}_1, \ldots, \mathbf{g}_P] \in \mathbb{C}^{M\times P}$ and $\mathbf{H} = [\mathbf{h}_1, \ldots, \mathbf{h}_P] \in \mathbb{C}^{N\times P}$, defined as

$$\mathbf{A} \approx \mathbf{G} \diamond \mathbf{H} = [\mathbf{g}_1 \otimes \mathbf{h}_1,\ \mathbf{g}_2 \otimes \mathbf{h}_2, \ldots,\ \mathbf{g}_P \otimes \mathbf{h}_P], \quad (1)$$

where $\diamond$ denotes the Khatri-Rao product and $\otimes$ denotes the Kronecker product.

To determine the factors of the Khatri-Rao product in the LS sense, one can employ the SVD of matrices formed from the individual columns of $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_P]$. Let $\mathbf{a}_p$ represent the p-th column of $\mathbf{A}$, where $p = 1, 2, \ldots, P$. The vector $\mathbf{a}_p$ can be reshaped into a matrix or tensor using an unvectorization operation, denoted as $\mathbf{A}_p^{(c)} = \mathrm{unvec}\{\mathbf{a}_p\} \in \mathbb{C}^{M\times N}$. Using the SVD of $\mathbf{A}_p^{(c)} = \mathrm{unvec}\{\mathbf{a}_p\}$, the best rank one approximation can be expressed as

$$\mathbf{A}_p^{(c)} = \mathrm{unvec}\{\mathbf{a}_p\} \approx \mathbf{u}_{p,1}\sigma_{p,1}\mathbf{v}_{p,1}^{\mathrm{H}}\ , \quad (2)$$

where $\mathbf{u}_{p,1}$ and $\mathbf{v}_{p,1}$ are the dominant left and right singular vectors, respectively, and $\sigma_{p,1}$ is the corresponding dominant singular value of $\mathbf{A}_p^{(c)}$. The SVD in (2) can be exploited to estimate $\mathbf{g}_p$ and $\mathbf{h}_p$ but these factors will be complex-valued scalar ambiguous; to see this, note that it can be expressed as

$$\hat{\mathbf{g}}_p = \mathbf{u}_1\alpha_p, \quad \hat{\mathbf{h}}_p = \mathbf{v}_1^*\beta_p, \quad (3)$$

where α_p and β_p are complex scalar values with constraint $\alpha_p\beta_p = \sigma_{p,1}$. The phase factor in this scaling ambiguity stems from the phase ambiguity of singular vectors [1]. Among this choice for α_p, β_p, we restrict ourselves to

$$\hat{\mathbf{g}}_p = \mathbf{u}_1\sqrt{\sigma_1}, \quad \hat{\mathbf{h}}_p = \mathbf{v}_1^*\sqrt{\sigma_1} \quad (4)$$

for simplicity. By repeating this process for each column of $\mathbf{A}$, the matrices $\mathbf{G}$ and $\mathbf{H}$ can be constructed, resulting in the best LS approximation of $\mathbf{A}$ by a Khatri-Rao product [7].

III. Extension of LS Khatri-Rao Factorization to Polynomial Matrices

In this section, we describe the extension of the Khatri-Rao factorisation to a polynomial matrix $\boldsymbol{A}(z) = [\boldsymbol{a}_1(a), \ldots, \boldsymbol{a}_P(z)]$, via analytic SVDs of $\mathrm{unvec}\{\boldsymbol{a}_p(z)\}$, $p = 1, \ldots, P$.

A. Existence of Analytic SVD

For an analytic polynomial matrix $\boldsymbol{A}(z) \in \mathbb{C}^{MN\times P}$, we seek an approximation via a Khatri-Rao product of two analytic polynomial matrices $\boldsymbol{G}(z) \in \mathbb{C}^{M\times P}$ and $\boldsymbol{H}(z) \in \mathbb{C}^{N\times P}$ akin to (1) as

$$\begin{aligned}\boldsymbol{A}(z) &\approx \boldsymbol{G}(z) \diamond \boldsymbol{H}(z) \\ &= [\boldsymbol{g}_1(z) \otimes \boldsymbol{h}_1(z),\ \ldots,\ \boldsymbol{g}_P(z) \otimes \boldsymbol{h}_P(z)],\end{aligned} \quad (5)$$

each $\boldsymbol{A}_p^{(c)}(z) = \mathrm{unvec}\{\boldsymbol{a}_p(z)\}, p = 1, \ldots, P$ must admit an analytic SVD, i.e. it must yield factors that are analytic and hence can be arbitrarily closely approximated by causal finite impulse response filters via shifts and truncations [18]. Such an analytic SVD exists with few exceptions [21]; these exceptions are avoided here because the elements of $\boldsymbol{A}_p^{(c)}(z)$ are estimated from finite data [22, 23]. Therefore we can find in all cases a factorisation

$$\boldsymbol{A}_p^{(c)}(z) = \boldsymbol{U}_p(z)\boldsymbol{\Sigma}(z)_p\boldsymbol{V}_p^{\mathrm{P}}(z)\ , \quad (6)$$

where $\{\cdot\}^{\mathrm{P}}$ is a para-Hermitian operator equivalent to Hermitian conjugate and time-reversal i.e. $\boldsymbol{V}_p^{\mathrm{P}}(z) = \{\boldsymbol{V}_p(1/z^*)\}^{\mathrm{H}}$ [24]. The matrices $\boldsymbol{U}_p(z) \in \mathbb{C}^{M\times M}$ and $\boldsymbol{V}_p(z) \in \mathbb{C}^{N\times N}$ hold in their columns the left- and right-singular vectors, respectively, and $\boldsymbol{\Sigma}_p(z) \in \mathbb{C}^{M\times N}$ contains analytic singular values on its diagonal. Both factors in (6) often have infinite order, but due to analyticity, they permit sufficiently accurate approximation through Laurent polynomials [18]. Both mth left- and right-singular vectors $\boldsymbol{u}_m(z)$ and $\boldsymbol{v}_m(z)$ are unique up to a common allpass factor, i.e. given an allpass function $\phi_m(z)$, $\boldsymbol{u}_m(z)\phi_m(z)$ and $\boldsymbol{v}_m(z)\phi_m(z)$ remain valid mth left- and right-singular vectors.

B. Analytic Least-Squares Khatri-Rao Factors

Via the analytic SVD factors of $\boldsymbol{A}_p^{(c)}(z)$ in (6), we can represent the frequency dependent version of (4) as

$$\boldsymbol{g}_p(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{u}_{p,1}(\mathrm{e}^{\mathrm{j}\Omega})\sqrt{\sigma_{p,1}(\mathrm{e}^{\mathrm{j}\Omega})}, \quad (7)$$

$$\boldsymbol{h}_p(\mathrm{e}^{\mathrm{j}\Omega}) = \boldsymbol{v}_{p,1}^*(\mathrm{e}^{\mathrm{j}\Omega})\sqrt{\sigma_{p,1}(\mathrm{e}^{\mathrm{j}\Omega})}. \quad (8)$$

As the analytic singular vectors exist [21], the analyticity of the factors in (7) and (8) relies on the whether the square root of the analytic singular values is analytic. Due to $\boldsymbol{A}(z)$ being estimated from finite data, $\sigma_{p,1}(\mathrm{e}^{\mathrm{j}\Omega})$ is strictly positive and $2\pi-$periodic for $p = 1, \ldots, P$, its square root will also be $2\pi-$periodic and therefore analytic [21, 25]. Similar to the analytic SVD factors, both $\boldsymbol{g}_p(z)$ and $\boldsymbol{h}_p(z)$ will generally be of infinite order. However, due to analyticity, the coefficients will decay at least as fast as exponential functions, therefore, a finite order Laurent polynomial approximation can be obtained via delay and truncation operations akin to analytic eigenvectors case for parahermitian matrices [18, 26].

The complex-valued scalar ambiguity translates into analytic functions, i.e. $\alpha_p(z)$ and $\beta(z)_p$ can be constrained such that $\alpha_p(z)\beta_p(z) = \sigma_{p,1}(z)$. Among the infinite solutions to this constraint, we here seek $\alpha_p(z) = \beta_p(z) = \sqrt{\sigma_{p,1}(z)}$ for $p = 1, \ldots, P$. Note that the square root of a polynomial can be determined via a Maclaurin series, with an example given in [27].

IV. DFT-Domain Implementation

A. SVD of $\boldsymbol{A}_p^{(c)}(z)$ in Sample Points

In order to determine LS factorisation of $\boldsymbol{A}(z)$ as a Khatri-Rao product, we may implement the ordinary matrix factorization within the sample points of $\boldsymbol{A}_p^{(c)}(z)$ akin to similar efforts in generalising polynomial matrix algebra [14, 17, 18, 28]. The conventional SVD computed independently in K sample points $\boldsymbol{A}_p^{(c)}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{A}_{p,k}^{(c)}$, with $\Omega_k, k = 0, \ldots, (K-1)$, produces

$$\mathbf{A}_{p,k} = \mathbf{U}_{p,k}\mathbf{\Sigma}_{p,k}\mathbf{V}_{p,k} \;, \tag{9}$$

where $\mathbf{U}_{p,k} = [\mathbf{u}_{p,1,k}, \ldots, \mathbf{u}_{p,M,k}]$ and $\mathbf{V}_{p,k} = [\mathbf{v}_{p,1,k}, \ldots, \mathbf{v}_{p,N,k}]$ are unitary matrices comprising the left- and right-singular vectors, respectively. The diagonal matrix $\mathbf{\Sigma}_{p,k} = \mathrm{diag}\{\sigma_{p,1,k}, \ldots, \sigma_{p,N,k}\}$ contains the bin-wise real-valued singular values such that $\sigma_{p,i,k} > \sigma_{p,i+1,k}\ \forall k, i = 1, \ldots, (M-1)$. Both the left- and right-singular vectors possess a phase ambiguity i.e., $\mathbf{u}_{m,k}\mathrm{e}^{\mathrm{j}\theta_{m,k}}$ and $\mathbf{v}_{m,k}\mathrm{e}^{\mathrm{j}\theta_{m,k}}$ are also a valid mth left- and right-singular vector of $\mathbf{A}_{p,k}$. Here we refer to the SVD in (9) as the bin-wise SVD in the kth frequency bin.

The bin-wise SVD in the kth bin relates to the samples of the analytic SVD at $z = \mathrm{e}^{\mathrm{j}\Omega_k}$ [23, 29] as

$$\boldsymbol{\Sigma}_p(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{\Sigma}_{p,k} \;, \tag{10}$$

$$\boldsymbol{U}_p(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{U}_{p,k}\mathbf{\Phi}_{p,k} \;, \tag{11}$$

$$\boldsymbol{V}_p(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{V}_k \begin{bmatrix} \mathbf{\Phi}_{p,k} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Phi}'_{p,k} \end{bmatrix} , \tag{12}$$

where, assuming $N \geq M$, $\mathbf{\Phi}_{p,k}$ is a diagonal matrix of phase shifts of size $M \times M$ and $\mathbf{\Phi}'_{p,k}$ is a unitary matrix of size $(N-M) \times (N-M)$. As a result, the singular vectors of the bin-wise SVD, computed within the sample points of $\boldsymbol{A}_p^{(c)}(z)$ on the unit circle, suffer from a loss of phase coherence between adjacent bins [18].

B. Sample Points of $\boldsymbol{g}_p(z)$ and $\boldsymbol{h}_p(z)$

The sample points $\boldsymbol{g}_p(\mathrm{e}^{\mathrm{j}\Omega_k}), \boldsymbol{h}_p(\mathrm{e}^{\mathrm{j}\Omega_k})$ can be determined via the relation between the sample points of the analytic SVD of $\boldsymbol{A}_p^{(c)}(z)$ at $z = \mathrm{e}^{\mathrm{j}\Omega_k}$ and the bin-wise SVD in (9) through (10)-(12) as

$$\boldsymbol{g}_p(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{u}_{p,1,k}\mathrm{e}^{\mathrm{j}\phi_{p,1,k}}\sqrt{\sigma_{p,1,k}} \tag{13}$$

$$\boldsymbol{h}_p(\mathrm{e}^{\mathrm{j}\Omega_k}) = \mathbf{v}^*_{p,1,k}\mathrm{e}^{-\mathrm{j}\phi_{p,1,k}}\sqrt{\sigma_{p,1,k}} \;. \tag{14}$$

This shows that the loss of phase coherence translates to the sample points of the columns of the LS Khatri-Rao factors i.e. $\boldsymbol{G}(z)$ and $\boldsymbol{H}(z)$. Therefore, unless the phase coherence is established via e.g. phase smoothing [18, 28], the solution may not be analytic, or at the very least the polynomial orders of the estimate of $\boldsymbol{G}(z)$ and $\boldsymbol{H}(z)$ may be significantly larger than necessary. Note that $\boldsymbol{g}_p(z)$ and $\boldsymbol{h}_p(z)$ are phase-coupled because both left- and right-singular vectors are phase-coupled via a common allpass function. Hence, phase smoothing is only required once for any column index p.

C. Algorithm and Convergence Metric

To estimate $\boldsymbol{G}(z)$ and $\boldsymbol{H}(z)$, the analytic SVD method of [28] is applied to determine only the dominant left- and right-singular vectors, i.e. $\boldsymbol{u}_{p,1}(z), \boldsymbol{v}_{p,1}(z)$ of $\boldsymbol{A}_p^{(c)}(z)$ for $p = 1, \ldots, P$ instead of the GSMD [30] and the GSBR2 [31] algorithms which provide approximate diagonalization and do not offer a reduced SVD computation of $\boldsymbol{A}_p^{(c)}(z)$.

Algorithm 1: LS Khatri-Rao Factorization of $\boldsymbol{A}(z)$

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Extract {u_{p,1}(z), σ_{p,1}(z), v_{p,1}(z)} for
  A_p^{(c)}, p = 1, ..., M via [17, 28, 32];
Set ζ_p = 1;
for p = 1 : P do
    Set K that exceeds order of σ_{p,1}(z);
    while ζ_p > ε do
        determine σ̃_{p,1}^{(K/2)}[τ] via (15);
        determine ζ_p via (16);
        σ̃_{p,1}(z) = Σ_τ σ̃_{p,1}^{(K/2)}[τ] z^{-τ};
        K ← 2K;
    end
    ĝ_p(z) = û_{p,1}(z) σ̃_{p,1}(z);
    ĥ_p(z) = v̂*_{p,1}(z) σ̃_{p,1}(z);
end
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The analytic SVD method requires invoking a phase smoothing algorithm [18] P times in total. For the dominant analytic singular value $\sigma_{p,1}(z)$ of $\boldsymbol{A}_p^{(c)}(z), p = 1, \ldots, P$, it is recommended to use the method in [32]. With analytic SVD factors computed for all P polynomial matrices, it remains to determine the square root of the analytic singular values with sufficient accuracy. To determine the square of para-Hermitian polynomials i.e. estimated analytic singular values, we compute the square in root in the DFT domain

$$\tilde{\sigma}_{p,1}(\mathrm{e}^{\mathrm{j}\Omega_k}) = \sqrt{\sigma_{p,1}(\mathrm{e}^{\mathrm{j}\Omega_k})}, k = 0, \ldots, (K-1) \tag{15}$$

at an increasing DFT size until the time-domain aliasing becomes negligible. For this, we define as metric

$$\zeta_p = \frac{\sum_\tau ||\tilde{\sigma}_{p,1}^{(K)}[\tau] - \tilde{\sigma}_{p,1}^{(K/2)}[\tau]||_\mathrm{F}^2}{\sum_\tau ||\tilde{\sigma}_{p,1}^{(K)}[\tau]||_\mathrm{F}^2}, \tag{16}$$

where $\tilde{\sigma}_{p,1}^{(K)}[\tau]$ and $\tilde{\sigma}_{p,1}^{(K/2)}[\tau]$ is the time-domain equivalent obtained through a K and $K/2$ point IDFT applied to $\{\tilde{\sigma}_{p,1}(\mathrm{e}^{\mathrm{j}\Omega_k}) | k = 0, \ldots, (K-1)\}$ and $\{\tilde{\sigma}_{p,1}(\mathrm{e}^{\mathrm{j}\Omega_{2k}}) | k = 0, \ldots, K/2-1\}$, respectively. A reasonably accurate estimate can be obtained based on the metric ζ_p which will decrease as K increases, and will eventually fall below a suitable threshold $\epsilon > 0$. This threshold defines when the DFT size is sufficiently large to minimise both time-domain aliasing and truncation errors to an acceptable level. The entire procedure for this LS Khatri-Rao factorisation is outlined in Algorithm 1.

V. Worked Example For LS Khatri-Rao Factorisation

In this worked example, we construct a $\boldsymbol{A}(z) \in \mathbb{C}^{4\times 2}$ of polynomial order 2 from known first order matrices $\boldsymbol{G}(z)$ and

$\boldsymbol{H}(z)$, given below with elements rounded to two decimals:

$$\boldsymbol{G}(z) = \begin{bmatrix} 0.98+0.98\mathrm{j} & -0.55-0.44\mathrm{j} \\ -0.27-1.19\mathrm{j} & -0.10+1.37\mathrm{j} \end{bmatrix} + \begin{bmatrix} -1.38-0.87\mathrm{j} & -1.89+0.25\mathrm{j} \\ -0.37+0.09\mathrm{j} & -2.94+0.41\mathrm{j} \end{bmatrix} z^{-1},$$

$$\boldsymbol{H}(z) = \begin{bmatrix} 0.65-1.77\mathrm{j} & 0.81+0.50\mathrm{j} \\ -0.16+0.44\mathrm{j} & 0.41+0.68\mathrm{j} \end{bmatrix} + \begin{bmatrix} -1.29-0.40\mathrm{j} & 0.65-0.37\mathrm{j} \\ 0.96+0.05\mathrm{j} & -0.41+0.26\mathrm{j} \end{bmatrix} z^{-1}.$$

The LS Khatri-Rao factors are estimated via Algorithm 1. The singular vectors are extracted via the phase smoothing algorithm at a DFT size of $K = 32$. The $\tilde{\sigma}_{p,1}(z), p = 1, \ldots, P$ are determined with a time-domain aliasing threshold of $\epsilon = 10^{-6}$. Subsequently, the accuracy of the estimated LS Khatri-Rao factors, i.e., $\hat{\boldsymbol{G}}(z)$ and $\hat{\boldsymbol{H}}(z)$, is determined, without applying any truncation, through a normalised difference w.r.t $\boldsymbol{A}(z)$ as

$$\xi = \frac{1}{K} \frac{\sum_{k=1}^{K} \|\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega_k}) - \hat{\boldsymbol{G}}(\mathrm{e}^{\mathrm{j}\Omega_k}) \diamond \hat{\boldsymbol{H}}(\mathrm{e}^{\mathrm{j}\Omega_k})\|_{\mathrm{F}}^2}{\sum_{k=1}^{K} \|\boldsymbol{A}(\mathrm{e}^{\mathrm{j}\Omega_k})\|_{\mathrm{F}}^2}, \quad (17)$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm of a matrix.

The estimated LS Khatri-Rao factors are of polynomial order is 31 after truncating trailing zero/coefficients and the error metric metric ξ is determined at a moderately high $K = 256$ to be of order 10^{-9}. This worked example indicates that the LS Khatri-Rao factorisation can be applied to polynomial matrices using the analytic SVD approach via phase smoothing.

VI. Broadband AoA From UPA Steering Matrix

Returning to the problem of AoA estimation for a broadband planar array outlined in Sec. I, we apply the proposed LS Khatri-Rao factorisation to the broadband steering matrix of a UPA. We simulate a scenario for two broadband sources firing from $\{\theta_1, \vartheta_1\} = \{-45^{\mathrm{o}}, 50^{\mathrm{o}}\}$ and $\{\theta_2, \vartheta_2\} = \{30^{\mathrm{o}}, -45^{\mathrm{o}}\}$ on an $M \times N$ UPA with $M = N = 8$ with 2D independent AoAs convention [34] as shown in Fig. 1. The angle ϑ represents the elevation but θ is not a direct measure of the azimuth. However, with basic trigonometry, azimuth can easily be determined from θ and ϑ. Given an ambiguity w.r.t. a wavefront either arriving from the front or the back of the array, i.e. either from positive or negative x-direction, we consider angular ranges $\theta, \vartheta \in [-\pi, \pi]$.

The broadband steering vectors are constructed based on windowed sinc function [33] of order 50 for $P = 2$ sources. The overall UPA steering matrices are known and can be represented as a Khatri-Rao product of steering matrices in azimuth and elevation directions, such that

$$\boldsymbol{S}(z) = \underbrace{\boldsymbol{S}_{\mathrm{az}}(z, \theta_{i=1,2})}_{\mathbb{C}^{8\times 2}} \diamond \underbrace{\boldsymbol{S}_{\mathrm{el}}(z, \vartheta_{i=1,2})}_{\mathbb{C}^{8\times 2}} .$$

The resultant $\boldsymbol{S}(z) \in \mathbb{C}^{64\times 2}$ effectively represents a UPA broadband steering matrix for the $P = 2$ sources at $\{\theta_1, \vartheta_1\} = \{-45^{\mathrm{o}}, 50^{\mathrm{o}}\}$ and $\{\theta_2, \vartheta_2\} = \{30^{\mathrm{o}}, -45^{\mathrm{o}}\}$, respectively. For the LS Khatri-Rao factorisation, Algorithm 1 is applied to $\boldsymbol{S}(z)$; this generates two independent steering matrices where

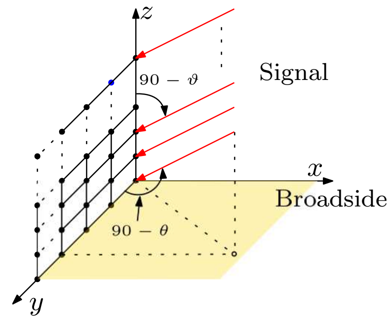

Fig. 1. Uniform planar array in yz-plane with independent 2D AoAs.

the pairing of AoAs are the same as the LS Khatri-Rao columns. Since only two sources are assumed, we only require the extraction of two rank-one terms, and the phase smoothing algorithm need to be invoked only twice.

With LS Khatri-Rao factors computed, we can apply the PMUSIC algorithm because the azimuth and elevation angles are now decoupled. Hence we capture the spatial MUSIC spectrum from the estimated LS Khatri-Rao factors and compare it against spatial MUSIC spectrum obtained directly from the ground truth factors i.e. $\boldsymbol{S}_{\mathrm{az}}(z, \theta_{i=1,2})$,and $\boldsymbol{S}_{\mathrm{el}}(z, \vartheta_{i=1,2})$. As shown in Fig. 2, the spectrum of the estimated LS Khatri-Rao factors closely aligns with the ground-truth Khatri-Rao factors for both directions, ensuring that the peak positions, and thus the spatial angles, remain consistent. Since each column of $\boldsymbol{S}(z)$ is decomposed into Kronecker factors, the angle pairing ambiguity is inherently resolved, eliminating the need for an explicit pairing operation as required in prior methods [9].

In practice, $\mathbf{S}[n]$, such that $\boldsymbol{S}(z) = \sum_n \mathbf{S}[n] z^{-n}$ or for short $\mathbf{S}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{S}(z)$, is often estimated from noisy sensor measurements. To model this effect, $\boldsymbol{S}(z)$ is perturbed using $\boldsymbol{E}(z)$, whose coefficients are sampled from a complex Gaussian distribution. The squared Frobenius norm of $\mathbf{E}[n] \circ\!\!-\!\!\bullet\, \boldsymbol{E}(z)$, calculated as $\sum_n \|\mathbf{E}[n]\|_{\mathrm{F}}^2$, is calibrated to $\frac{1}{10}$ of squared Frobenius norm of $\mathbf{S}[n]$. Subsequently, the LS Khatri-Rao factorisation is applied to the perturbed $\boldsymbol{S}(z)$, and the resulting PMUSIC spectrum is presented alongside the non-perturbed LS Khatri-Rao factorisation results in Fig. 2. It is evident from these results that the estimated AoAs remain very accurate when compared to the ground-truth AoAs despite the introduced perturbations.

VII. Conclusion

In this paper, we have introduced the LS Khatri-Rao factorisation of a polynomial matrix, leveraging the existence of an analytic SVD for polynomial matrices. By determining the dominant analytic singular vectors and singular value for the unvectorized representation of each column of the given polynomial matrix, LS Khatri–Rao factors are derived in a manner analogous to their counterparts in conventional matrix analysis. While we have not yet assessed nor compared the computational complexity due to the lack of a suitable benchmark, we have demonstrated that the proposed algorithm is particularly applicable to broadband UPAs, where it enables

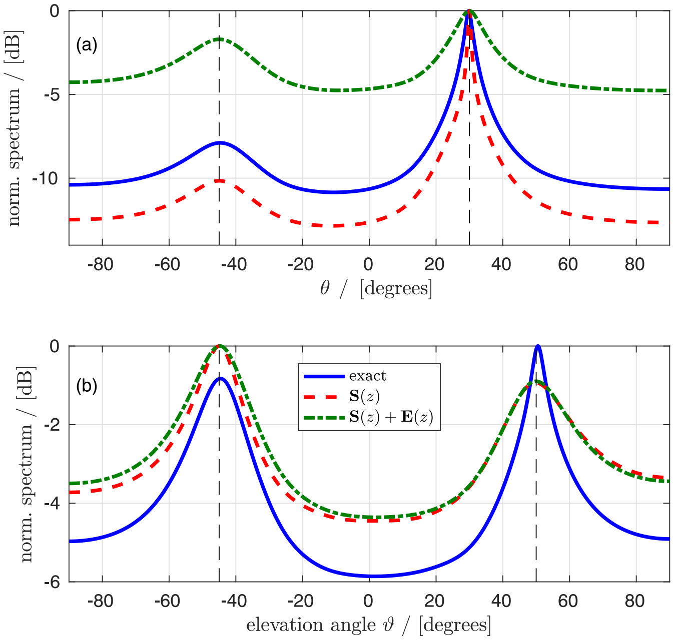

Fig. 2. Polynomial spatial MUSIC normalized spectrum for (a) θ, and (b) ϑ (or elevation angle), LS Khatri-Rao factorisation of $\boldsymbol{S}(z)$ and LS Khatri-Rao factorisation of $\boldsymbol{S}(z)+\boldsymbol{E}(z)$ perturbed by a term $\boldsymbol{E}(z)$ with $\sum_n \|\mathbf{E}[n]\|_{\mathrm{F}}^2/(\sum_n \|\mathbf{S}[n]\|_{\mathrm{F}}^2)=\frac{1}{10}$.

the decoupling of azimuth and elevation directions for broadband sources while facilitating automatic angle pairing.

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