Approach to beamforming minimizing the signal power estimation error

Esa Ollila
Aalto University
Espoo, Finland
esa.ollila@aalto.fi

Xavier Mestre
CTTC
Castelldefels, Spain
xavier.mestre@cttc.cat

Elias Raninen
Nokia Bell Labs
Espoo, Finland
elias.raninen@nokia-bell-labs.com

***Abstract*—We study the properties of beamformers, particularly Capon and MMSE beamformers, in their ability to obtain the true signal power of the signal of interest (SOI). A curious feature of these beamformers is their tendency to either overestimate or underestimate the signal power. Consequently, they are not asymptotically unbiased (as the sample size approaches to infinity). To address this issue, we propose to shrink the Capon beamformer by finding a scaling factor that minimizes the mean squared error (MSE) of the signal power estimate. The new beamformer, referred to as the Capon$^+$ beamformer, is evaluated against the Capon and MMSE beamformers in terms of bias, signal power MSE, and signal waveform MSE. The Capon$^+$ beamformer demonstrates a superior balance in both signal power and waveform estimation while exhibiting minimal bias, which approaches zero as the sample size increases.**

***Index Terms*—beamforming, Capon, signal power estimation, shrinkage**

I. Introduction

Spatial filtering (beamforming) is a widely used multi antenna technique that allows recovering signals that are contaminated by interference, clutter or colored noise. In some cases, recovering the signal up to a certain scalar factor is sufficient, as it does not significantly impact the output signal quality. However, in other scenarios, it is crucial to preserve the exact power of the signal as received at each individual antenna in the array. This is particularly important in communication applications without training symbols, where accurate signal recovery is necessary, along with the correct scaling factor to prevent constellation expansion or compression. A widely used linear spatial filter is Capon's [1] beamformer, also known as the *minimum power distortionless response (MPDR)* beamformer [2, Sec. 6.2.4] or *minimum variance distortionless response (MVDR) beamformer*. The weights of the beamformer depend on the array covariance matrix, composed of interference plus noise covariance (INCM) and the power and steering vector of the signal of interest (SOI). However, it is well known that Capon's beamformer is not performing well in signal power estimation. The spatial spectrum of the adaptive Capon's filter tends to underestimate the power in small samples [3], while overestimate in large samples as is shown in this work. Another popular spatial filter, signal estimation minimum mean squared error (MMSE) [2, Sec. 6.2.2] beamformer, on the other, underestimates the signal power in large sample sizes (*cf.* Sect. 2). Consequently, neither of these beamformers is asymptotically unbiased as the sample size approaches infinity.

To mitigate this issue, we propose a shrinkage-based modification to Capon's beamformer, where we determine a scaling factor for Capon's beamformer that minimizes the signal power mean squared error (MSE). The resulting beamformer, referred to as Capon$^+$ beamformer, provides a more balanced trade-off between power and signal waveform estimation compared to both the Capon and the MMSE beamformers.

We consider an array of M sensors, where the array data (snapshots) follows a linear model:

$$\mathbf{x}(t) = \mathbf{a}s(t) + \mathbf{e}(t),\ t = 1, \dots, T,$$

where $\mathbf{a} \in \mathbb{C}^M$ is the steering vector for the SOI, $\mathbf{e}(t) \in \mathbb{C}^M$ is a random vector consisting of interference and noise, $s(t)$ is the signal waveform of the SOI, and T denotes the number of snapshots. The steering vector $\mathbf{a}$ is dependent on the location parameters (e.g., the direction of arrival (DOA) θ of the SOI). The array covariance matrix $\boldsymbol{\Sigma} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}(t)^\mathsf{H}]$ has the form:

$$\boldsymbol{\Sigma} = \gamma \mathbf{a}\mathbf{a}^\mathsf{H} + \mathbf{Q}, \tag{1}$$

where $\gamma = \mathbb{E}[|s(t)|^2]$ is the SOI power and $\mathbf{Q} = \mathbb{E}[\mathbf{e}(t)\mathbf{e}(t)^\mathsf{H}]$ is the INCM due to interfering sources and noise.

Let $\mathbf{w}$ denote the *beamformer weight* for the SOI. The output of the beamformer, $\hat{s}(t) = \mathbf{w}^\mathsf{H}\mathbf{x}(t)$, serves as the estimate of the signal waveform $s(t)$, and

$$\hat{\gamma} = \frac{1}{T}\sum_{t=1}^{T} |\hat{s}(t)|^2 = \frac{1}{T}\sum_{t=1}^{T} |\mathbf{w}^\mathsf{H}\mathbf{x}(t)|^2 \tag{2}$$

serves as the *estimate of the SOI power* γ. With fixed (non-random) $\mathbf{w}$, the (expected) beamformer output power is

$$\mathbb{E}[\hat{\gamma}] = \mathbb{E}[|\hat{s}(t)|^2] = \mathbf{w}^\mathsf{H}\boldsymbol{\Sigma}\mathbf{w}. \tag{3}$$

The *Capon* beamformer minimizes (3) subject to the constraint that the SOI is passed undistorted:

$$\min_{\mathbf{w}} \mathbf{w}^\mathsf{H}\boldsymbol{\Sigma}\mathbf{w} \quad \text{subject to } \mathbf{w}^\mathsf{H}\mathbf{a} = 1. \tag{4}$$

The optimum beamformer weight that solves (4) is [2]:

$$\mathbf{w}_{\text{Cap}} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{a}}{\mathbf{a}^\mathsf{H}\boldsymbol{\Sigma}^{-1}\mathbf{a}} = \frac{\mathbf{Q}^{-1}\mathbf{a}}{\mathbf{a}^\mathsf{H}\mathbf{Q}^{-1}\mathbf{a}}. \tag{5}$$

The corresponding output power is given by

$$\gamma_{\text{Cap}} = \mathbb{E}[|\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^2] = \mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{\Sigma}\mathbf{w}_{\text{Cap}} = \frac{1}{\mathbf{a}^{\mathsf{H}}\mathbf{\Sigma}^{-1}\mathbf{a}}, \quad (6)$$

and thus $\mathbf{w}_{\text{Cap}} = \gamma_{\text{Cap}}\mathbf{\Sigma}^{-1}\mathbf{a}$. The bias of the power estimator

$$\hat{\gamma}_{\text{Cap}} = \frac{1}{T}\sum_{t=1}^{T}|\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^2. \quad (7)$$

based on the Capon beamformer is $\mathsf{B}(\hat{\gamma}_{\text{Cap}}) = \mathbb{E}[\hat{\gamma}_{\text{Cap}}] - \gamma = (\mathbf{a}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a})^{-1}$, which is *always* positive, i.e., the Capon beamformer overestimates the signal power. This result is a direct consequence of [4, Lemma 1]. The power estimator $\hat{\gamma}_{\text{Cap}}$ is not asymptotically unbiased either (when $T \to \infty$).

Based on the above, we seek an optimal scaling constant $\beta > 0$ for the shrinkage estimator of the form

$$\mathbf{w}_{\beta} = \beta\mathbf{w}_{\text{Cap}}. \quad (8)$$

We determine a scaling factor $\alpha = \beta^2$ that minimizes the MSE of the associated signal power estimator. We call the resulting beamformer as the Capon$^+$ beamformer. We analyze the performance of the Capon, MMSE, and Capon$^+$ beamformers in estimating or maintaining the presumed signal power. Specifically, the beamformers are evaluated based on their bias and signal power estimation MSE as well as the signal waveform estimation accuracy.

The paper has connections to previous works. For example, [5] proposed robust enhancements of the MMSE beamformer (9) taking into account imperfect knowledge of SOI signal power level. A linear combination of Capon beamformer and the conventional delay-and-sum beamformer was considered in [6], wherein the estimation of the optimal weighting coefficients that minimize the signal estimation MSE was considered under the random matrix regime (RMT) and assuming that signal power is known. RMT based beamformers have also been developed in [7]. Robust beamforming techniques [8], [9] address uncertainties in the array steering vector, often caused by calibration errors or inaccuracies in the SOI location parameters. However, robust beamformers often exhibit suboptimal performance in signal power estimation.

II. Motivational Example

The MMSE beamformer is defined by

$$\min_{\mathbf{w}} \left\{ \text{MSE}(\mathbf{w}) = \mathbb{E}[|s(t) - \mathbf{w}^{\mathsf{H}}\mathbf{x}(t)|^2] \right\}. \quad (9)$$

The solution to (9) is

$$\mathbf{w}_{\text{MMSE}} = \gamma\mathbf{\Sigma}^{-1}\mathbf{a} = \frac{\gamma\mathbf{Q}^{-1}\mathbf{a}}{1 + \gamma\mathbf{a}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a}}. \quad (10)$$

Hence the Capon and MMSE beamformers are identical up to a scale with relation

$$\mathbf{w}_{\text{MMSE}} = \frac{\gamma}{\gamma_{\text{Cap}}}\mathbf{w}_{\text{Cap}}. \quad (11)$$

The power of the MMSE beamformer on the other hand is

$$\gamma_{\text{MMSE}} = \mathbb{E}[|\mathbf{w}_{\text{MMSE}}^{\mathsf{H}}\mathbf{x}(t)|^2] = \gamma \cdot \frac{\gamma}{\gamma_{\text{Cap}}}, \quad (12)$$

which follows from (11) and (6). Since $\gamma/\gamma_{\text{Cap}} < 1$, the bias of the MMSE beamformer is *always* negative. This is precisely the opposite situation compared to the Capon beamformer. Moreover, since $\gamma/\gamma_{\text{Cap}} \not\to 1$ as $T \to \infty$, it follows from (12) that the MMSE beamformer is not asymptotically unbiased either.

Based on the above, a shrinkage estimator of the form (8) with $\beta \in [\gamma/\gamma_{\text{Cap}}, 1]$ can strike a balance between the Capon and the MMSE beamformer. This is illustrated next.

In Figure 1 we display the relative bias, $(\hat{\gamma} - \gamma)/\gamma$, the empirical signal estimation normalized MSE (NMSE),

$$\text{SE-NMSE}_T = \sum_{t=1}^{T}|\hat{s}(t) - s(t)|^2 \Big/ \sum_{t=1}^{T}|s(t)|^2, \quad (13)$$

and the signal power estimation NMSE, defined as

$$\text{SP-NMSE}_T = \frac{(\hat{\gamma} - \gamma)^2}{\gamma^2}.$$

All metrics are averaged over 15000 MC trials. As beamformer weights we used the true weights (e.g., $\mathbf{w}_{\text{MMSE}}$, $\mathbf{w}_{\text{Cap}}$ or $\mathbf{w}_{\text{Cap}^+}$), i.e., $\mathbf{Q}$ and γ are assumed to be known.

Simulation setting: The array is a Uniform Linear Array (ULA) with $M = 25$ antennas and sources are narrowband and farfield. The steering vector is defined as

$$\mathbf{a} = \mathbf{a}(\theta) \triangleq (1, e^{-\jmath \cdot 1 \cdot \frac{2\pi d}{\lambda}\sin\theta}, \ldots, e^{-\jmath \cdot (M-1) \cdot \frac{2\pi d}{\lambda}\sin\theta})^{\top},$$

where λ is the wavelength, d is the element spacing between the sensors and $\theta \in \Theta = [-\pi/2, \pi/2)$ is the direction-of-arrival (DOA) of the SOI in radians. We assume $d = \lambda/2$ sensor spacing. There are 4 independent circular complex Gaussian sources: the SOI has DOA $-45.02°$ while the three interfering sources arrive from DOAs $-30.02°$, $-20.02°$, $-3°$, respectively, and having signal powers that are, respectively, 2, 4, and 6 dB lower than the power of the SOI. The noise is white Gaussian with unit variance and the number of snapshots is $T = 60$. The results are shown in Figure 1.

As can be noted from the top panel of Figure 1, the power estimation bias of the Capon and MMSE beamformers are significant in low SNR. The former having significant overestimation while the latter underestimation bias. The middle panel shows that the Capon$^+$ beamformer (defined in Section III) has slightly worse performance to MMSE beamformer in signal waveform estimation, but this is compensated by its nearly zero bias and much better signal power estimation displayed in the bottom panel. This is not surprising since the Capon$^+$ beamformer is designed to yield the minimum MSE in signal power estimation.

III. Optimal Signal Power Beamformer

We have shown that the Capon and the MMSE beamformer either overshoot or undershoot signal power estimation. This implies that a shrinkage estimator of the form (8) using $\beta \in [\gamma/\gamma_{\text{Cap}}, 1]$ can strike a balance between the Capon and the MMSE beamformer. We derive such an optimal beamformer next. The proofs are available in extended arXiv submission of this work [10] due to page limits.

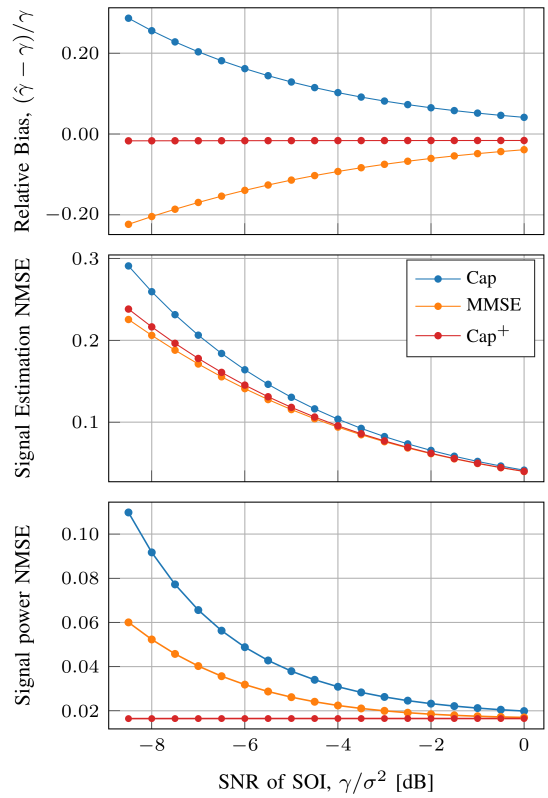

Fig. 1: *Top:* Relative bias of SOI power estimate. *Middle:* signal estimation NMSE of SOI. *Bottom:* Power estimation NMSE of SOI. There are 3 interfering sources with signal powers of -2, -4, and -6 dB relative to the SOI.

Consider the power estimator $\hat{\gamma}$ in (2) for some fixed (known) $\mathbf{w}$. The MSE of $\hat{\gamma}$ is

$$\mathsf{MSE}(\hat{\gamma}) = \mathsf{var}(\hat{\gamma}) + [\mathsf{B}(\hat{\gamma})]^2, \tag{14}$$

where $\mathsf{B}(\hat{\gamma}) = \mathbb{E}[\hat{\gamma}] - \gamma$ and $\mathsf{var}(\hat{\gamma}) = \mathbb{E}[(\hat{\gamma} - \mathbb{E}[\hat{\gamma}])^2]$ are the bias and the variance of $\hat{\gamma}$, respectively. Let $\mathcal{CN}_M(\mathbf{0}, \mathbf{\Sigma})$ denote the M-variate circular complex normal distribution with zero mean and positive definite Hermitian $M \times M$ covariance matrix $\mathbf{\Sigma}$. We have the following result.

Lemma 1. *For fixed (known)* $\mathbf{w}$*, the variance of* $\hat{\gamma}$ *in* (2) *is*

$$\mathsf{var}(\hat{\gamma}) = \frac{\mathbb{E}[|\mathbf{w}^{\mathsf{H}}\mathbf{x}(t)|^4] - (\mathbf{w}^{\mathsf{H}}\mathbf{\Sigma}\mathbf{w})^2}{T}. \tag{15}$$

Furthermore, if $\mathbf{x}(t) \overset{iid}{\sim} \mathcal{CN}_M(\mathbf{0}, \mathbf{\Sigma})$, $t = 1, \ldots, T$, *then*

$$\mathsf{var}(\hat{\gamma}) = \frac{1}{T}(\mathbf{w}^{\mathsf{H}}\mathbf{\Sigma}\mathbf{w})^2. \tag{16}$$

Consider an estimator of the form

$$\hat{\gamma} = \alpha\hat{\gamma}_{\text{Cap}} = \frac{1}{T}\sum_{t=1}^{T}|\mathbf{w}_{\beta}^{\mathsf{H}}\mathbf{x}(t)|^2, \tag{17}$$

where $\mathbf{w}_{\beta}$ is defined in (8) and $\alpha = \beta^2$. We next determine the coefficient α such that the beamformer output power of the shrinked beamformer optimally reflects the true power level.

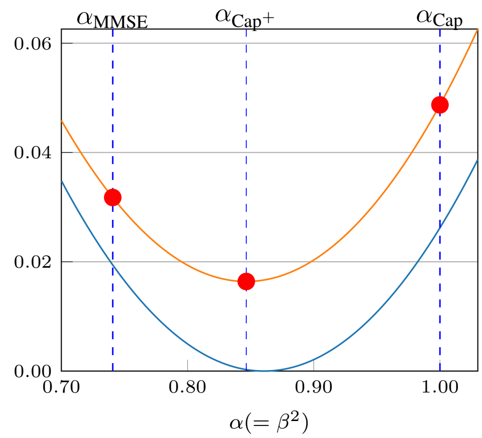

Fig. 2: The NMSE (yellow) and the relative squared bias (blue) of $\hat{\gamma} = \alpha\hat{\gamma}_{\text{Cap}}$ as a function of α when SOI has -6 dB SNR.

Theorem 1. *The value* α *minimizing the MSE* $\mathbb{E}[(\alpha\hat{\gamma}_{\text{Cap}} - \gamma)^2]$ *is*

$$\alpha_{\text{o}} = \frac{T\gamma_{\text{Cap}}\gamma}{\mathbb{E}[|\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4] + (T-1)\gamma_{\text{Cap}}^2}, \tag{18}$$

where γ_{Cap} *is defined in* (6)*. The minimum MSE obtained by* $\hat{\gamma}_{\text{Cap}^+} = \alpha_{\text{o}}\hat{\gamma}_{\text{Cap}}$ *is*

$$\mathsf{MSE}_{\min} = \mathbb{E}[(\alpha_{\text{o}}\hat{\gamma}_{\text{Cap}} - \gamma)^2] = \gamma^2\frac{\mathsf{var}(\hat{\gamma}_{\text{Cap}})}{\mathbb{E}[\hat{\gamma}_{\text{Cap}}^2]}, \tag{19}$$

where $\mathsf{var}(\hat{\gamma}_{\text{Cap}}) = (\mathbb{E}[|\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4] - \gamma_{\text{Cap}}^2)/T$*. Furthermore, if* $\mathbf{x}(t) \overset{iid}{\sim} \mathcal{CN}_M(\mathbf{0}, \mathbf{\Sigma})$*, then*

$$\alpha_{\text{o}} = \frac{\gamma}{\gamma_{\text{Cap}}} \cdot \frac{T}{T+1} \quad \textit{and} \quad \mathsf{MSE}_{\min} = \frac{\gamma^2}{T+1}. \tag{20}$$

Hence, in the Gaussian case, the optimal signal power estimator $\hat{\gamma}_{\text{Cap}^+} = \alpha_{\text{o}}\hat{\gamma}_{\text{Cap}}$ has an expected value of

$$\mathbb{E}[\hat{\gamma}_{\text{Cap}^+}] = \alpha_{\text{o}}\gamma_{\text{Cap}} = \gamma\frac{T}{T+1}$$

and $\mathsf{B}(\hat{\gamma}_{\text{Cap}^+}) = -\gamma/(T+1)$, which is negligible already for moderate T. This can be seen in the top panel of Figure 1. In addition, the minimum MSE in (19) is not dependent on the SNR, i.e., the beamformer is able to maintain accurate power balancing even at low SNR scenarios. This feature is visible in the bottom panel of Figure 1 that shows the signal power estimation NMSE. Note that the NMSE is heavily increasing for the MMSE and the Capon beamformer as SNR decreases while it remains constant for the Capon$^+$ beamformer. In terms of the signal waveform estimation NMSE, we can notice that the MMSE beamformer has the best performance, yet Capon$^+$ is inferior to it only in low SNR cases.

Figure 2 shows the NMSE of $\hat{\gamma}_{\text{Cap}^+} = \alpha\hat{\gamma}_{\text{Cap}}$ as a function of α and the relative squared bias, $[\mathsf{B}(\hat{\gamma})/\gamma]^2$, for an SOI with SNR of -6 dB. The NMSE is minimized at α_{o} from (20), marked by the dotted vertical line. Also shown are $\alpha = 1$ (Capon) and $\alpha = \gamma^2/\gamma_{\text{Cap}}^2$ (MMSE). Notably, Capon and

MMSE estimators exhibit large biases, while Capon$^+$ achieves near-zero bias. Capon$^+$ also reduces the signal power NMSE by about 3x compared to Capon and 2x compared to MMSE beamformer. Thus, the proposed beamformer accurately preserves the true signal power at the output with minimal error while simultaneously achieving the lowest possible MSE.

IV. Simulation studies

It is important to highlight that the signal power γ is usually unknown or the assumed signal power is inaccurate. Also the covariance matrices $\boldsymbol{\Sigma}$ and $\mathbf{Q}$ are in practice unknown and are typically replaced by their estimates. Thus it is important to propose an adaptive Capon$^+$ beamformer for the cases when γ is unknown and/or INCM $\mathbf{Q}$ is unknown, and these quantities needs to be estimated. The simulation setting is the same as earlier, described in Section II, with one distinction: the signals are no longer Gaussian random signals, but 8-PSK modulated random signals with fixed constant squared amplitude, $\gamma_k = |s_k(t)|^2$, $k = 1, \ldots, 4$. The codes are available at https://github.com/esollila/Capon_plus

A. Scenario A: INCM $\mathbf{Q}$ *is known,* γ *is unknown*

Scenario A typically corresponds to a multi-antenna radar application where the receiver has access to secondary data without the presence of the SOI. In this scenario, estimating the power of the signal of interest, along with its DOA, can be used to determine the target's position. The scenario implies full knowledge of $\mathbf{w}_{\text{Cap}}$, but a need to estimate γ. Note that $\gamma = \gamma_{\text{Cap}} - (\mathbf{a}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a})^{-1}$, yielding the following estimate which can be described as a debiased form of the Capon power estimator:

$$\hat{\gamma}_{\text{deb}} = \max(\hat{\gamma}_{\text{Cap}} - (\mathbf{a}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a})^{-1}, 0), \tag{21}$$

where $\hat{\gamma}_{\text{Cap}}$ is defined by (7) and $\max(\cdot, 0)$ is used to guarantee that the estimate remains positive. As shrinkage constant for Capon$^+$ we may now use

$$\hat{\alpha}_{\text{Cap}^+} = \frac{T\hat{\gamma}_{\text{Cap}}\hat{\gamma}_{\text{deb}}}{\frac{1}{T}\sum_{t=1}^{T}|\mathbf{w}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4 + (T-1)\hat{\gamma}_{\text{Cap}}^2}, \tag{22}$$

where $\hat{\gamma}_{\text{deb}}$ is given in (21). The Capon$^+$ beamformer weight is then simply $\mathbf{w}_{\text{Cap}^+} = \sqrt{\hat{\alpha}_{\text{Cap}^+}}\mathbf{w}_{\text{Cap}}$ as $\beta^2 = \alpha$. When implementing the MMSE beamformer in (10), we replace the true γ by its estimate $\hat{\gamma}_{\text{deb}}$ and use the latter form in (10) which gives

$$\mathbf{w}_{\text{MMSE}} = \frac{\hat{\gamma}_{\text{deb}}\mathbf{Q}^{-1}\mathbf{a}}{1 + \hat{\gamma}_{\text{deb}}\mathbf{a}^{\mathsf{H}}\mathbf{Q}^{-1}\mathbf{a}}$$

as the MMSE beamformer weight.

The results are shown in Figure 3 and they can be compared to Figure 1. First, we notice that the estimate $\hat{\alpha}_{\text{Cap}^+}$ works well, and the obtained signal power $\hat{\gamma}_{\text{Cap}^+}$ is essentially unbiased for all SNR levels. This is clearly not the case for the MMSE and the Capon beamformer. Also the signal power NMSE is the best among the methods as expected. In terms of signal estimation NMSE, the Capon$^+$ and the MMSE beamformers have similar performance except at very low SNR. This indicates that Capon$^+$ beamformer provides a better alternative to the MMSE beamformer in practical settings.

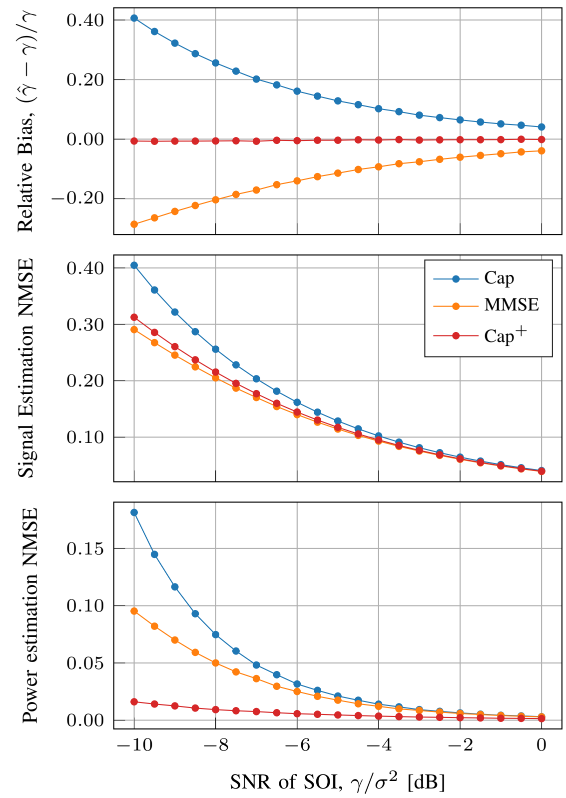

Fig. 3: Results for scenario A. The sample length $T = 60$.

B. Scenario B: $\mathbf{Q}$ *is unknown,* γ *is known*

Scenario B could represent a communication setting where the power of the signal of interest remains stable over time, allowing for accurate estimation, while spatial interference fluctuates due to the varying activity of other sources. It implies that we need to use an adaptive Capon beamformer, where we estimate the unknown $\boldsymbol{\Sigma}$ by the *sample covariance matrix (SCM)* $\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}(t)\mathbf{x}(t)^{\mathsf{H}}$, and estimate the Capon beamformer by

$$\hat{\mathbf{w}}_{\text{Cap}} = \hat{\hat{\gamma}}_{\text{Cap}}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{a} \quad \text{with} \quad \hat{\hat{\gamma}}_{\text{Cap}} = (\mathbf{a}^{\mathsf{H}}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{a})^{-1}. \tag{23}$$

Note that the covariance matrix and the weight vector are estimated from the same snapshots. This is slightly different from the sample matrix inversion (SMI) adaptive beamformer [11], which uses an independent secondary data set of only interference and noise samples to estimate the INCM $\mathbf{Q}$. Note that $\hat{\hat{\gamma}}_{\text{Cap}}$ is not the same estimator as $\hat{\gamma}_{\text{Cap}}$ as the latter uses true $\mathbf{w}_{\text{Cap}}$ while the former uses adaptive weight vector $\hat{\mathbf{w}}_{\text{Cap}}$ in (23). For large sample lengths, however, they are equivalent. Since the signal power γ of the SOI is known, the Capon$^+$ beamformer is defined as $\hat{\mathbf{w}}_{\text{Cap}^+} = \sqrt{\hat{\alpha}_{\text{Cap}^+}}\hat{\mathbf{w}}_{\text{Cap}}$ with the shrinkage constant estimated using

$$\hat{\alpha}_{\text{Cap}^+} = \frac{T\hat{\hat{\gamma}}_{\text{Cap}}\gamma}{\frac{1}{T}\sum_{t=1}^{T}|\hat{\mathbf{w}}_{\text{Cap}}^{\mathsf{H}}\mathbf{x}(t)|^4 + (T-1)\hat{\hat{\gamma}}_{\text{Cap}}^2}. \tag{24}$$

Note also that the shrinkage constants in (22) and (24) are different due to different assumptions in Scenarios A and B.

Further note that an adaptive MMSE beamformer in scenario B is simply $\hat{\mathbf{w}}_{\text{MMSE}} = \gamma \hat{\mathbf{\Sigma}}^{-1} \mathbf{a}$.

When $\mathbf{\Sigma}$ is estimated, both data stationarity and a larger number of snapshots than the $T = 60$ used in Figure 1 are required before we can observe that beamformer's performance aligns with the asymptotic scenario. We present results for two different sample sizes: $T = 200$ and $T = 500$. Figure 4 shows the results when the sample size to estimate the covariance matrix is $T = 200$. As can be noted, for small sample length, the MMSE and Capon beamformer's performance is far from desired or even expected. We can notice that in the higher SNR, the adaptive Capon beamformer has underestimation bias (instead of overestimation bias as in the asymptotic case) while the adaptive MMSE beamformer has overestimation bias (instead of underestimation bias). This strong reverse effect of bias causes the MMSE beamformer to perform worse than the other two beamformers in terms of signal estimation NMSE for SNR > -5 dB. For $T = 500$, the large sample effects starts to kick in and the beamformers start to behave more similarly to what is observed in Figure 1 and Figure 3. Notably, the Capon^{+} beamformer provides solid performance throughout.

V. Conclusions

We considered beamformers of the form $\mathbf{w}_\beta = \beta \mathbf{w}_{\text{Cap}}$. The Capon (resp. MMSE beamformer) power estimates $\hat{\gamma}_{\text{Cap}}$ (resp. $\hat{\gamma}_{\text{MMSE}}$) can exhibit a large positive (resp. negative) bias and high signal power MSE at low SNR, indicating a suboptimal bias-variance tradeoff.The proposed Capon^{+} beamformer achieves minimal signal power MSE and comparable signal waveform MSE as the optimal MMSE beamformer, offering a better balance between power and waveform estimation.

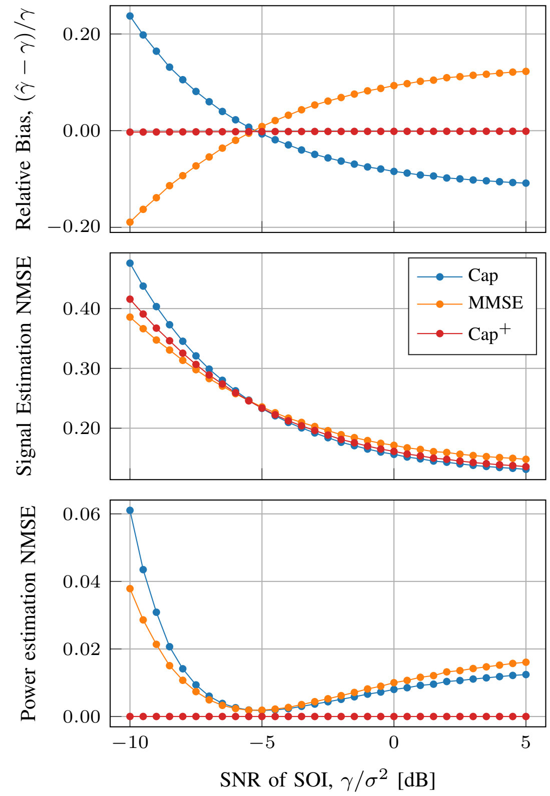

Fig. 4: Results for scenario B. The sample length $T = 200$.

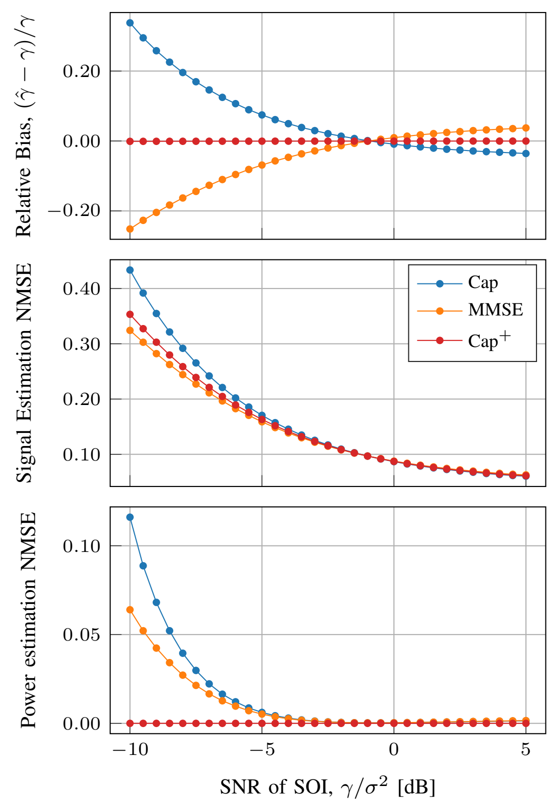

Fig. 5: Results for scenario B. The sample length $T = 500$.

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