On the Performance of Sensing Matrix Design-Based Alternating Minimization Approach for Compressive Sensing

Baghdad Hadji[1,2], Abdeldjalil Aïssa-El-Bey[3], Lamya Fergani[2] and Mustapha Djeddou[4]
[1] Faculty of Technology, Saïda University, 20000 Saïda, Algeria
Email:hadjb.ebm@gmail.com
[2] LISIC Laboratory,University of Science and Technology Houari Boumediene, Algiers, Algeria
Email: wissalle.2017@gmail.com
[3]IMT Atlantique, UMR CNRS 6285 Lab-STICC, Brest F-29238, France
Email:abdeldjalil.aissaelbey@imt-atlantique.fr
[4] Ecole Nationale Polytechnique,El Harrach, 16000 Algiers, Frères Oudek street 10, Algeria
Email:mustapha.djeddou@g.enp.edu.dz

***Abstract*—The iterative alternating minimization approach has gained increasing attention for the development of sensing matrix design algorithms. However, relying solely on the mutual coherence index as a single evaluation metric does not guarantee optimal sensing matrix performance. To enhance Compressed Sensing (CS) recovery accuracy, it is crucial to simultaneously minimize multiple coherence metrics (μ_{mx}, μ_{ave}, and μ_{all}), particularly by leveraging Equiangular Tight Frame (ETF) properties. In this paper, we propose a novel iterative alternating minimization method for designing an optimized sensing matrix that reduces mutual coherence values. Our approach incorporates a simple yet effective shrinkage function, which enables the approximation of an ideal ETF structure during the updating phase while maintaining computational efficiency. Experimental results demonstrate that the sensing matrix designed using the proposed algorithm achieves superior performance in terms of both signal reconstruction accuracy and coherence reduction, outperforming existing methods.**

***Index Terms*—Compressed sensing framework, Equiangular Tight Frame (ETF),orthogonal matching pursuit (OMP), Greedy algorithms, iterative alternating minimization approach.**

I. Introduction

In recent years, compressed sensing (CS) [1], [2] has emerged as a powerful mathematical framework for efficiently acquiring and processing signals while overcoming the limitations of the Shannon-Nyquist sampling theorem. CS enables the recovery of an unknown signal $\boldsymbol{x} \in \mathbb{R}^{N\times 1}$ from a few measurements $\boldsymbol{y} \in \mathbb{R}^{M\times 1}$ by exploiting its sparse representation:

$$\boldsymbol{y} = \boldsymbol{\Phi x} = \underbrace{\boldsymbol{\Phi\Psi}}_{\boldsymbol{D}} \boldsymbol{s}, \tag{1}$$

where $\boldsymbol{s} \in \mathbb{R}^{L\times 1}$ is a k-sparse representation of $\boldsymbol{x}$ in a given sparsifying dictionary $\boldsymbol{\Psi} \in \mathbb{R}^{N\times L}$, while $\boldsymbol{\Phi} \in \mathbb{R}^{M\times N}$ is the sensing matrix, and $\boldsymbol{D}$ is equivalent matrix. Recovering the k nonzero components of $\boldsymbol{s}$ requires solving the following optimization problem:

$$\arg\min \|\boldsymbol{s}\|_0 \quad s.t \;\; \boldsymbol{y} = \boldsymbol{\Phi\Psi s}. \tag{2}$$

However, this ℓ_0-norm minimization is NP-hard and computationally intractable in general [3]. To circumvent this issue, CS frameworks commonly employ ℓ_1-norm minimization and greedy algorithms, which offer computationally feasible sparse reconstructions via convex relaxation techniques [4], [5]. The performance of CS reconstruction algorithms heavily depends on the quality of the sensing matrix $\boldsymbol{\Phi}$, which must satisfy key properties, such as spark, Restricted Isometry Property (RIP), and mutual coherence [6]. However, using the spark and RIP are not tractable in practice to design the sensing matrix [7]. Furthermore, applying the mutual coherence index alone in the designing of the sensing matrix does not ensure a highly CS-based systems performance. Therefore, the optimal sensing matrix should be granting an orthogonal equivalent matrix version in CS theory whatever the known dictionary [8]. To benefit the mutual coherence as criteria in practice, the robust optimal sensing matrix must reduce the correlation constant between any distinct pair of the corresponding Gram matrix $\boldsymbol{G}$ ($\boldsymbol{G} = \boldsymbol{\Psi}^H\boldsymbol{\Phi}^H\boldsymbol{\Phi\Psi}$) by updating its version according to the equiangular tight frame (ETF) properties [5]–[9]. In addition, the optimal sensing matrix can be obtained through an iterative alternating approach, incorporating ETF properties to solve the Frobenius norm minimization problem for the difference between the Gram matrix and the target Gram matrix $\boldsymbol{G_t}$:

$$\underset{\boldsymbol{\Phi}, \boldsymbol{G_t} \in \mathcal{H}_{\mu_{welch}}}{\arg\min} \left\|\boldsymbol{G_t} - \boldsymbol{\Psi}^H\boldsymbol{\Phi}^H\boldsymbol{\Phi\Psi}\right\|_F^2 \tag{3}$$

where $\mathcal{H}_{\mu_{welch}}$ represents a convex set containing ideal ETF elements [10]

$$\mathcal{H}_{\mu_{welch}} = \Big\{ \boldsymbol{G_t} \in \mathbb{R}^{L\times L} : \boldsymbol{G_t} = \boldsymbol{G_t}^H, \\ \operatorname{diag}(\boldsymbol{G_t}) = 1, \max_{i\neq j} |\boldsymbol{G_t}(i,j)| \leq \mu_{welch} \Big\} \tag{4}$$

whilst $\mu_{welch} \triangleq \sqrt{\frac{L-M}{M(L-1)}}$ defining the Welch bound. The optimization problem in (3) presents two primary challenges:

1) Updating $\boldsymbol{G_t}$ to closely approximate an ETF structure,
2) Deriving the optimal sensing matrix $\boldsymbol{\Phi}$ that minimizes mutual coherence.

In [5], authors propose a minimization solution using a shrinkage method to iteratively update $\boldsymbol{G_t}$, reducing the t-averaged mutual coherence and optimizing $\boldsymbol{\Phi}$. However, this alternating minimization method incurs a high computational cost, and the thresholding approach in the update phase fails to yield ideal ETF elements due to persistently high off-diagonal values in $\boldsymbol{G_t}$. This negatively impacts both the maximum and average mutual coherence (μ_{mx} and μ_{ave}), ultimately degrading the performance of reconstruction algorithms. An alternative iterative method introduced in [9] aims to design an optimal sensing matrix using a power method to obtain the largest absolute eigenvalue instead of performing full eigenvalue decomposition. However, this approach fails in cases where eigenvalues are negative or when dealing with complex-valued systems. To address computational complexity issues and improve recovery performance, [7] proposes a new thresholding mechanism within the shrinkage method to refine $\boldsymbol{G_t}$ and reduce mutual coherence values. Nevertheless, a major limitation of this approach is the absence of a closed-form mathematical expression for determining an optimal threshold across different CS applications. In this paper, we introduce an iterative alternating minimization algorithm for sensing matrix design. Our method leverages a classical shrinkage approach with a simple yet effective thresholding mechanism to update $\boldsymbol{G_t}$ according to ETF properties, resulting in improved sensing matrix quality and superior signal recovery performance, as confirmed by our simulation results.

II. CS-BASED THEORY BACKGROUND

This section introduces fundamental concepts related to CS, including ETF properties and mutual coherence metrics. The maximum mutual coherence of any matrix $\boldsymbol{D} \in \mathbb{R}^{M\times N}$ is defined as the largest absolute normalized inner product between any two distinct columns:

$$\mu_{mx}(\boldsymbol{D}) = \max_{i\neq j, 1\leq i,i\leq L} \left\{ \frac{\left|\boldsymbol{d}_i^T \boldsymbol{d}_j\right|}{\|\boldsymbol{d}_i\|_2^2 \cdot \|\boldsymbol{d}_j\|_2^2} \right\} \tag{5}$$

This metric quantifies the highest degree of correlation between different columns of $\boldsymbol{D}$. Therefore, the design of the sensing matrix based on μ_{mx} cannot guarantee a better accuracy recovery result for any recovery algorithms. To provide a more comprehensive evaluation of the sensing matrix quality, additional coherence metrics are defined based on the Gram matrix $\boldsymbol{G} = \boldsymbol{D}^H\boldsymbol{D}$ such as the maximum, averaged, and global mutual coherence values (μ_{mx}, μ_{ave} and μ_{all}) of the off-diagonal elements of $\boldsymbol{G}$ as expressed in the following definitions

$$\mu_{mx} = \max_{i\neq j} |\tilde{g}_{ij}| \tag{6}$$

$$\mu_{ave} = \frac{\sum_{i\neq j} \left(|\tilde{g}_{ij} \geq t|\right) |\tilde{g}_{ij}|}{\sum_{i\neq j} \tilde{g}_{ij} \geq t} \tag{7}$$

$$\mu_{all} = \sum_{i\neq j} \tilde{g}_{ij}^2 \tag{8}$$

where t is the threshold value proposed by Elad [5] to reduce the mutual coherence where $\mu_{ave} \geq t$. Whereas, $\tilde{g}_{ij} = \tilde{\boldsymbol{d}}_i^T \tilde{\boldsymbol{d}}_j$ is the entry at the position of row i and column j in $\tilde{\boldsymbol{G}}$ ($\tilde{\boldsymbol{G}} = \tilde{\boldsymbol{D}}^H\tilde{\boldsymbol{D}}$), and $\tilde{\boldsymbol{D}}$ is column-normalized version of $\boldsymbol{D}$. The correlation between column pairs should ideally reach the Welch bound, which represents the theoretical limit of incoherence and ensures high recovery performance. To reduce the different mutual coherence values simultaneously to the lower bound, the ETF properties must be exploited in the design of the sensing matrix with respect to the known dictionary. Also, approximating ETF is the main phase to solve the optimization problem in (3). Consequently, we briefly highlight the concept and principal characteristics of a frame in the design of the optimal sensing matrix as follows :

The matrix $\boldsymbol{D} = [\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n] \in \mathbb{R}^{m\times n}$ is called a frame with $m \ll n$, if there exist two constants $0 < \alpha \leq \beta \leq +\infty$ such that

$$\alpha \|\boldsymbol{v}\|_2 \leq \left\|\boldsymbol{D}^T\boldsymbol{v}\right\|_2 \leq \beta \|\boldsymbol{v}\|_2, \forall \boldsymbol{v} \in \mathbb{R}^m \tag{9}$$

where α and β are the lower and the upper bound of frames respectively [11]. If $\alpha = \beta$ in (9), the frame $\boldsymbol{D}$ is called α-tight frame, and when $\alpha = \beta = 1$, is called a Parseval frame. Similarly, the overcomplete dictionary $\boldsymbol{D}$ is called ETF, if the following conditions are satisfied

- Each column has a unit norm : $\|\boldsymbol{d_i}\|_2$ for $i = 1, \ldots, n$.
- The columns are equiangular. For some nonnegative δ, we get $|\langle \boldsymbol{d_i d_j}\rangle| = \delta$ when $i \neq j,\ i, j = 1, \ldots, n$.
- The columns form a tight frame. That is, $\boldsymbol{D}\boldsymbol{D}^H = \left(\frac{n}{m}\right)\boldsymbol{I_m}$, where $\boldsymbol{I_m}$ is identity matrix of size $m \times m$.

III. THE PROPOSED SENSING MATRIX DESIGNING

In this work, we propose an iterative alternating minimization approach that combines a classical shrinkage method for updating $\boldsymbol{G_t}$ and an alternating projection technique to design an optimal sensing matrix that simultaneously minimizes mutual coherence values.

Theorem 1: (see [8]): Let $\boldsymbol{\Psi} = \boldsymbol{U_\Psi}\left[\boldsymbol{\Sigma_\Psi}\quad \boldsymbol{0}\right]\boldsymbol{V_\Psi^H}$ be the singular value decomposition of $\boldsymbol{\Psi}$, where $\boldsymbol{U_\Psi} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{V_\Psi} \in \mathbb{R}^{n\times n}$ are unitary matrices. If rank($\boldsymbol{\Psi}$) $= m < n$, then $\boldsymbol{\Sigma_\Psi}$ contains m singular values with $\sigma_1 \geq \sigma_2 \ldots \geq \sigma_m$. Suppose that $\tilde{\boldsymbol{G}}_{\boldsymbol{t}} \in \mathcal{H}_{\mu_{welch}}$, and if $\boldsymbol{\Theta} = \boldsymbol{V_\Psi^H}\tilde{\boldsymbol{G}}_{\boldsymbol{t}}\boldsymbol{V_\Psi}$ is positive semidefinite matrix, then $\boldsymbol{\Theta} = \boldsymbol{X_\Theta A_\Theta X_\Theta^H}$ is the eigendecomposition of $\boldsymbol{\Theta}$. The optimal $\boldsymbol{\Phi_{opt}}$ can be find by the following solution to solve the problem in (3)

$$\boldsymbol{\Phi_{opt}} = \boldsymbol{\Lambda}_{\boldsymbol{\Theta}}^{\frac{1}{2}}\boldsymbol{P}^H \left[\boldsymbol{\Sigma_\Psi^{-1}}\quad \boldsymbol{0}\right]^H \boldsymbol{U_\Psi^H} \tag{10}$$

where $\boldsymbol{\Lambda_\Theta} \in \mathbb{R}^{m\times m}$ is diagonal matrix that contain m maximum eigenvalues of $\boldsymbol{\Theta}$, whereas $\boldsymbol{P} \in \mathbb{R}^{n\times m}$ denotes the first m columns of $\boldsymbol{X_\Theta}$ corresponding to the top m eigenvalues. For the proof of Theorem 1, see the appendix in [8].

Our goal is to iteratively optimize $\boldsymbol{\Phi}$ while refining $\boldsymbol{G_t}$ via classical shrinkage. Algorithm 1 details the procedure. As summarized in Algorithm 1, we initialize $\boldsymbol{\Phi}$ to a random matrix where $\boldsymbol{\Psi}$ is introduced as a given sparsifying dictionary to obtain the equivalent dictionary $\boldsymbol{D} = \boldsymbol{\Phi\Psi}$. Afterward, we normalize the columns in $\boldsymbol{D}$ during each iteration to produce $\tilde{\boldsymbol{D}}$ that is used in the next step to get the gram matrix $\tilde{\boldsymbol{G}}$. As the optimal sensing design is only achieved through the ETF properties, we must update $\boldsymbol{G}$ to be close to the corresponding ETF designed by projecting the Gram matrix elements $\tilde{g}_{ij}$ on $\mathcal{H}_{\mu_{welch}}$ to have unit diagonal elements and reduce the off-diagonals. For this purpose, we choose Welch bound as the known threshold to get the updated version of a $\tilde{\boldsymbol{G}}_{\boldsymbol{t}}$ as

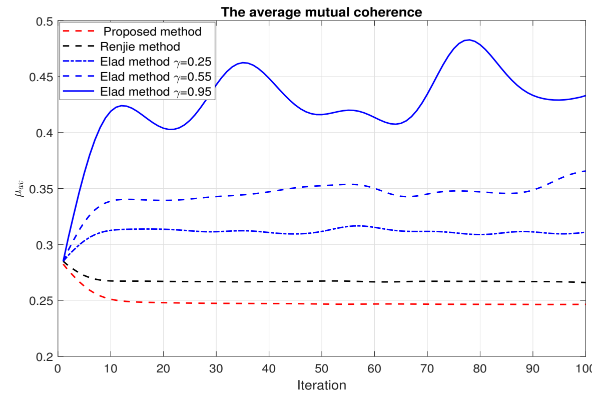
$$\forall\, i, j \;\; i \neq j : \tilde{\boldsymbol{G}}_{\boldsymbol{t}}(i,j) = \begin{cases} \tilde{g}_{ij} & \text{abs}(\tilde{g}_{ij}) < \mu_{welch} \\ \text{sign}(\tilde{g}_{ij}) & \text{otherwise} \end{cases} \tag{11}$$

Algorithm 1 Iterative Alternating Minimization for Sensing Matrix Design

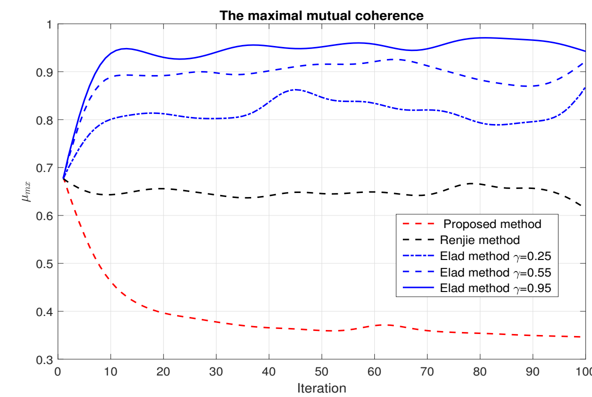
Input: sparsifying basis $\boldsymbol{\Psi}$ with SVD decomposition $\boldsymbol{\Psi} = \boldsymbol{U}_{\boldsymbol{\Psi}} \left[\boldsymbol{\Sigma}_{\boldsymbol{\Psi}} \;\; \boldsymbol{0}\right] \boldsymbol{V}_{\boldsymbol{\Psi}}^{H}$,
Welch bound μ_{welch}, number of iterations *Iter*
Output: Optimized sensing matrix $\hat{\boldsymbol{\Phi}}$
Initialization: Set $\boldsymbol{\Phi}$ as a random matrix.
for k to *Iter* **do**
1) $\boldsymbol{\Phi}_{(\boldsymbol{k})} \leftarrow \boldsymbol{\Phi}$
2) Compute the equivalent matrix $\boldsymbol{D} = \boldsymbol{\Phi}_{\boldsymbol{k}}\boldsymbol{\Psi}$
3) Compute the Gram matrix $\tilde{\boldsymbol{G}} = \tilde{\boldsymbol{D}}^{H}\tilde{\boldsymbol{D}}$ ($\tilde{\boldsymbol{D}}$ is normalization version of $\boldsymbol{D}$)
4) Update $\tilde{\boldsymbol{G}}$ to obtain $\tilde{\boldsymbol{G}}_{\boldsymbol{t}}$ using (11)
5) Compute the positive semidefinite matrix $\boldsymbol{\Theta} = \boldsymbol{V}_{\boldsymbol{\Psi}}^{H}\tilde{\boldsymbol{G}}_{\boldsymbol{t}}\boldsymbol{V}_{\boldsymbol{\Psi}}$
6) Apply eigenvalue decomposition to obtain $\boldsymbol{\Theta} = \boldsymbol{X}_{\boldsymbol{\Theta}}\boldsymbol{\Lambda}_{\boldsymbol{\Theta}}\boldsymbol{X}_{\boldsymbol{\Theta}}^{H}$
• Find $\boldsymbol{\Lambda}_{\boldsymbol{\Theta}} \in \mathbb{R}^{m\times m}$ including m maximum eigenvalues of $\boldsymbol{A}_{\boldsymbol{\Theta}}$
• Find $\boldsymbol{P} \in \mathbb{R}^{n\times m}$ containing the first columns of $\boldsymbol{X}_{\boldsymbol{\Theta}}$
7) Update $\boldsymbol{\Phi}_{(\boldsymbol{k+1})}$ using (10)
end for

IV. SIMULATION RESULTS

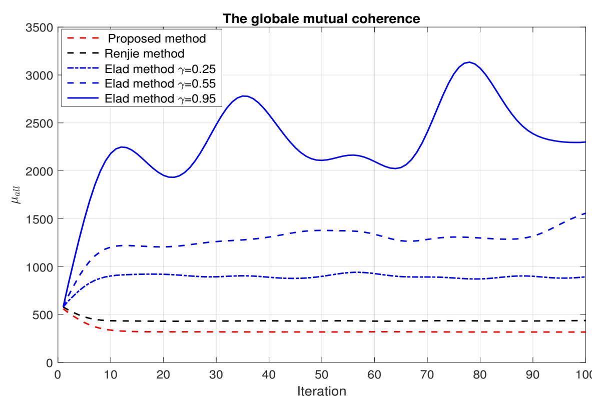
In this section, we provide a set of simulations to elucidate the performance of the proposed method and compare it with other sensing matrix designs, such as Elad's method [5] and Renjie's method [7]. During the carried out simulations,, the parameter t for Elad's method is fixed to 0.2 with three different values of down-scaling factor γ: $\gamma_1 = 0.25, \gamma_2 = 0.55, \gamma_3 = 0.95$, whearas c is set to 0.01 for Renjie's method [7]. In Fig. 1, mutual coherence values evolution as a function of outer iteration numbers are presented, where the given dictionary matrix $\boldsymbol{\Psi} \in \mathbb{R}^{80\times 120}$ is a random Gaussian matrix and $\boldsymbol{\Phi} \in \mathbb{R}^{28\times 80}$ is generated randomly as the initial matrix in the beginning of each design method. As shown in Fig.1 (a), (b), and (c), the mutual coherence values obtained by the proposed method are reached to the lower bound values with a simple choice of the threshold in the updating phase, which means that the proposed method has a better sensing matrix design in terms of the decreasing of the mutual coherence values (μ_{mx}, μ_{ave} and μ_{all}) simultaneously.



(a)



(b)



(c)

Fig. 1. Evolution results of: (**a**) the t-averaged mutual coherence μ_{ave}, (**b**) the maximal coherence μ_{mx}, and (**c**) the global mutual coherence μ_{all}, all versus iteration number for an 28×80 random matrix $\boldsymbol{\Phi}$ as initial matrix and an 80×120 dictionary matrix $\boldsymbol{\Psi}$ with Gaussian distribution.

Fig. 2 illustrates the histogram of the absolute off-diagonal entries of the optimized Gram matrix where we can see that the histogram obtained by the proposed method has a shift towards the left (or origin) with small values of correlation constant between pairwise of columns in the updated $\boldsymbol{G}_{\boldsymbol{t}}$ matrix compared to the other design method.

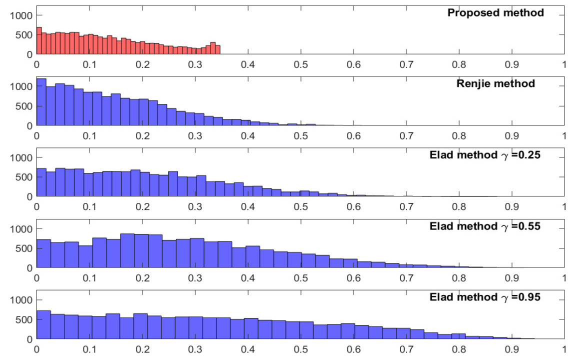
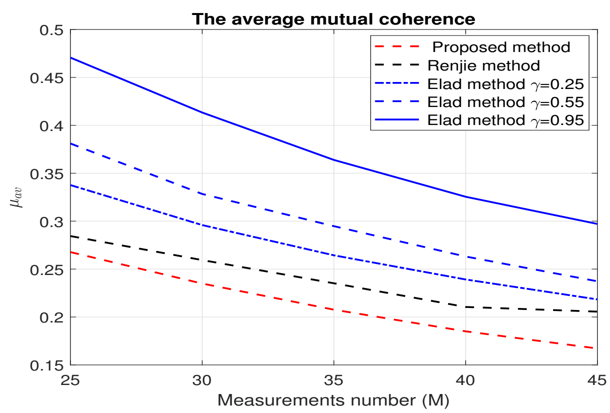
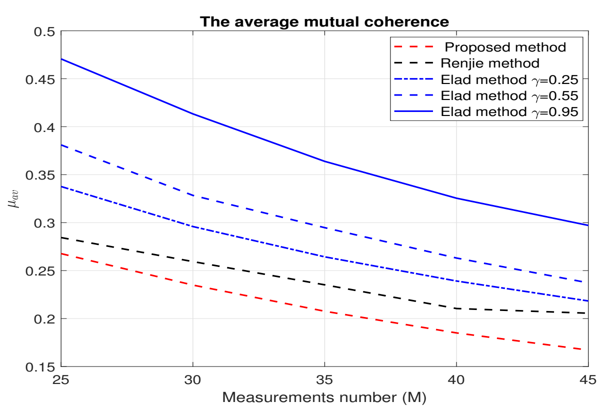


Fig. 2. Histogram of the absolute value of the off-diagonal entries of the updated $\boldsymbol{G_t}$ matrix.

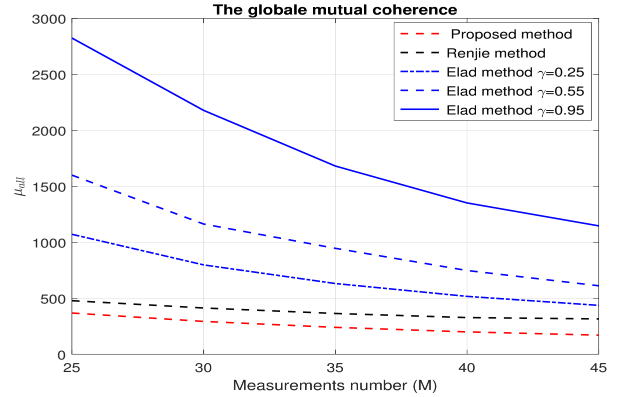
As the measurement numbers have a relationship with the Welch bound, we evaluate the evolution of mutual coherence indexes by varying the measurement numbers of the sensing matrix. From Fig.3 (a), (b), and (c), the mutual coherence values corresponding to each method decrease with high measurement numbers, and we can note that mutual coherence value results confirm that the proposed method outperforms the other methods.



(a)



(b)



(c)

Fig. 3. Evolution results of: (**a**) the t-averaged mutual coherence μ_{ave}, (**b**) the maximal coherence μ_{mx}, and (**c**) the global mutual coherence μ_{all}, all versus measurement numbers.

For testing the recovery accuracy performance, we evaluate the performance of the designed sensing matrix using the reconstruction error defined as : $\|\hat{\boldsymbol{x}} - \boldsymbol{x}\|_2 \,/\, \|\boldsymbol{x}\|_2^2$ where $\hat{\boldsymbol{x}}$ is the recovered signal by Orthogonal Matching Pursuit (OMP) algorithm by varying the sparsity K and generating k-sparse signals $\boldsymbol{s}$ from a standard normal distribution $(0, 1)$ with i.i.d. elements. Fig.4 depicts the reconstruction error results as a function of sparsity levels.

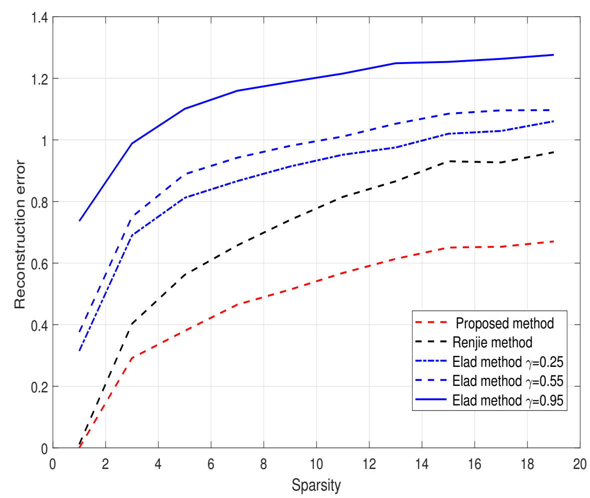


Fig. 4. Comparison results of reconstruction error v.s. different sparsity levels K.

According Fig. 4 , we can note that the proposed algorithm can achieve better recovery accuracy than other methods along all sparsity levels. For a fixed sparsity level $K = 8$, and fixed dimension $L = 120$, $N = 80$, Fig. 5 represents errors results by varying the measurement numbers M. From this figure, recovery accuracy results obtained by the proposed sensing matrix design via OMP algorithm is better than the other results. Under different SNR values from 5 to $50dB$, we evaluate the recovery error of the AWGN model as illustrated in Fig.6 with fixed dimension $M = 28$ and sparsity level $K = 8$. As demonstrated, recovery error results acquired by the proposed method are excellent compared to the other results.

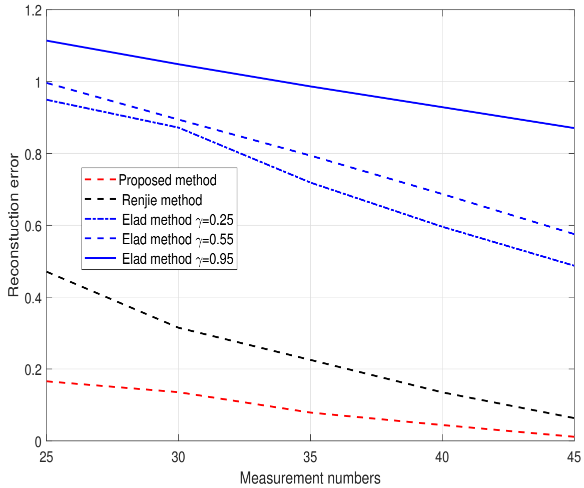


Fig. 5. Comparison results of reconstruction error results v.s. different measurement numbers M

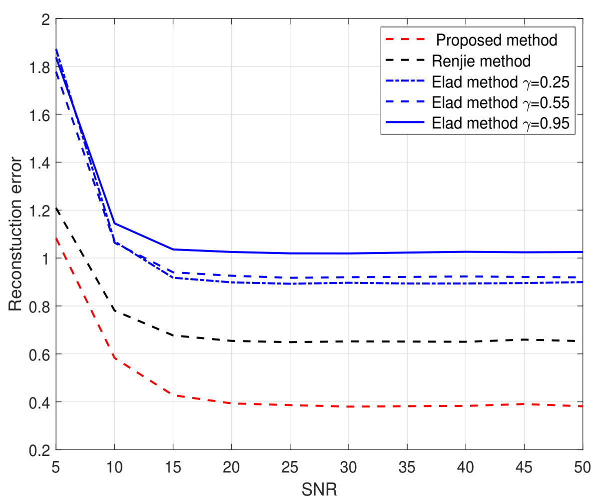


Fig. 6. Comparison results of reconstruction error results v.s. SNR.

V. CONCLUSION

In this paper, we proposed and analyzed an iterative alternating minimization approach for optimizing sensing matrix design. Our method simultaneously minimizes the mutual coherence values (μ_{mx} , μ_{ave} , and μ_{all}) to enhance CS recovery performance. Furthermore, the proposed algorithm incorporates a simple yet effective shrinkage function in the update phase to approximate ETF properties, enabling the design of robust sensing matrices applicable to various CS scenarios. Simulation results confirm that our method significantly reduces mutual coherence indices while improving signal reconstruction accuracy compared to existing approaches. Future work will explore extensions to non-Gaussian dictionary structures, adaptive thresholding strategies, and real-time implementations for large-scale applications.

REFERENCES

[1] D. L. Donoho, "Compressed sensing," *IEEE Transactions on information theory*, vol. 52, no. 4, pp. 1289–1306, 2006.

[2] E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Transactions on information theory*, vol. 52, no. 2, pp. 489–509, 2006.

[3] S. Friedland, Q. Li, and D. Schonfeld, "Compressive sensing of sparse tensors," *IEEE Transactions on Image Processing*, vol. 23, no. 10, pp. 4438–4447, 2014.

[4] A. L. Pilastri and J. M. R. Tavares, "Reconstruction algorithms in compressive sensing: An overview," in *11th edition of the Doctoral Symposium in Informatics Engineering (DSIE-16)*, 2016.

[5] M. Elad, "Optimized projections for compressed sensing," *IEEE Transactions on Signal Processing*, vol. 55, no. 12, pp. 5695–5702, 2007.

[6] C. Rusu and N. González-Prelcic, "Optimized compressed sensing via incoherent frames designed by convex optimization," *arXiv preprint arXiv:1507.02454*, 2015.

[7] R. Yi, C. Cui, B. Wu, and Y. Gong, "A new method of measurement matrix optimization for compressed sensing based on alternating minimization," *Mathematics*, vol. 9, no. 4, p. 329, 2021.

[8] B. Hadji, A. Aïssa-El-Bey, L. Fergani, and M. Djeddou, "Joint hybrid precoding and combining design based multi-stage compressed sensing approach for mmwave mimo channel estimation," *IEEE Access*, vol. 11, pp. 112398–112413, 2023.

[9] T. Hong, H. Bai, S. Li, and Z. Zhu, "An efficient algorithm for designing projection matrix in compressive sensing based on alternating optimization," *Signal Processing*, vol. 125, pp. 9–20, 2016.

[10] J. A. Tropp, I. S. Dhillon, R. W. Heath, and T. Strohmer, "Designing structured tight frames via an alternating projection method," *IEEE Transactions on information theory*, vol. 51, no. 1, pp. 188–209, 2005.

[11] E. V. Tsiligianni, L. P. Kondi, and A. K. Katsaggelos, "Construction of incoherent unit norm tight frames with application to compressed sensing," *IEEE Transactions on Information Theory*, vol. 60, no. 4, pp. 2319–2330, 2014.