Invariant CFAR Detection in Gaussian Disturbance with Kronecker Covariance Structure

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***Abstract*—This paper studies the adaptive detection of range-spread targets in a Gaussian environment, assuming a Kronecker-product structure for the disturbance covariance matrix. Invoking the principle of invariance, we identify a transformation group that significantly reduces the dimensionality of nuisance parameters, ensuring the constant false alarm rate (CFAR) property for all invariant statistics. A maximal invariant (MI) is derived, providing the foundation for new CFAR detectors that are invariant tests functionally depending on the MI. At the stage of detector design, two adaptive detectors are devised: the former employs the two-step strategy and incorporates the Kronecker maximum likelihood estimate based on secondary data, while the latter is the one-step generalized likelihood ratio test realized via an alternating-optimization algorithm. Both are invariant tests, and thus their CFAR properties with respect to the Kronecker covariance matrix are naturally guaranteed. Numerical results demonstrate the superior detection performance and robust CFAR behavior of both the detectors compared to conventional methods designed for the unstructured case.**

***Index Terms*—Adaptive detection, Kronecker structure, invariant theory, CFAR.**

I. INTRODUCTION

Reliable adaptive detection of radar targets generally demands accurate estimation of disturbance covariance matrices [1]–[3]. However, in practical scenarios, the available training data are often limited, leading to performance degradation of conventional detectors designed under the assumption of a completely unknown covariance matrix [4]. Indeed, depending on operating scenarios and radar system characteristics, specific structures on the disturbance covariance matrix may be identified. Incorporating such prior knowledge into detector design has emerged as an effective strategy to enhance estimation efficiency and detection capability [5].

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The Kronecker-product structure has recently gained popularity in multidimensional signal processing, owing to its extensive application in areas such as space-time adaptive processing for classic side-looking array geometry [6], [7], polarimetric array radar [8], [9], and multiple-input-multiple-output radar [10]. Estimators that effectively exploit this structure can reduce the amount of training data required to achieve satisfactory inference accuracy. However, the constant false alarm rate (CFAR) property of plug-in detectors, which incorporate these structured estimators directly into conventional architectures, still requires further verification and is not always guaranteed particularly when regularization techniques are employed (e.g., [8]).

Invariant detection theory [11], [12] provides a principled framework for designing CFAR detectors. Specifically, enforcing invariance under a suitably identified group of transformations can reduce the number of nuisance parameters without compromising essential information for target detection task. Once the interplay between invariance and CFARity is well-established, invariant tests often automatically guarantee the CFAR property. Existing studies have explored related frameworks for unstructured covariance [1], [13], [14], persymmetric [15], [16], and block-diagonal structures [12], [17], [18], but invariant detection under the Kronecker covariance structure has not yet been explored in the open literature.

Motivated by the above reason, this paper conducts an invariance analysis for range-spread target detection under Gaussian disturbance with a Kronecker-structured covariance matrix. We derive a maximal invariant (MI) and an induced MI, and show that any invariant test ensures the CFAR property. Furthermore, two invariant detectors are derived whose CFAR properties are theoretically guaranteed. Numerical examples indicate their significant performance improvements over existing approaches.

The remainder of the paper is organized as follows. Section II formulates the hypotheses testing problem and its canonical form. Section III details the invariance analysis. Section IV focuses on the design of novel detectors, also providing an algorithm for the maximum likelihood estimate (MLE) under the H_1 hypothesis. Simulation results appear in Section V, followed by conclusions in Section VI.

II. Problem Formulation

Consider a radar system equipped with N_2 receive channels transmitting a coherent burst of N_1 pulses. In each pulse repetition interval, the radar acquires H range samples to adequately cover the target range interval, yielding data snapshots $\boldsymbol{x}_i^0 \in \mathbb{C}^{N\times 1}$, $i = 1, \ldots, H$, where $N = N_1 N_2$. Thus, the range-distributed target detection problem addressed in this study can be formulated in terms of the binary hypothesis test

$$\begin{cases} \mathrm{H}_1 : \boldsymbol{x}_i^0 = \alpha_i(\boldsymbol{p}_1 \otimes \boldsymbol{p}_2) + \boldsymbol{d}_i^0, \\ \mathrm{H}_0 : \boldsymbol{x}_i^0 = \boldsymbol{d}_i^0, \end{cases} \quad i = 1, \ldots, H, \tag{1}$$

where $\boldsymbol{p}_1 \in \mathbb{C}^{N_1\times 1}$ and $\boldsymbol{p}_2 \in \mathbb{C}^{N_2\times 1}$ represent unit-norm temporal and spatial steering vectors, respectively; $\alpha_i \in \mathbb{C}, i = 1, \ldots, H$, are unknown deterministic amplitude parameters; and $\boldsymbol{d}_i^0 \in \mathbb{C}^N, i = 1, \ldots, H$, are disturbance random vectors independent and identically distributed as $\mathcal{CN}(\boldsymbol{0}, \boldsymbol{M}_1^0 \otimes \boldsymbol{M}_2^0)$, with $\boldsymbol{M}_1^0 \in \mathbb{H}_{++}^{N_1}$ and $\boldsymbol{M}_2^0 \in \mathbb{H}_{++}^{N_2}$. As customary, it is also assumed that a set of secondary (training) data $\boldsymbol{x}_i^0, i = H+1, \ldots, H+K$, is available. These secondary data contain no useful target signal but share the same disturbance characteristics as the primary data.

To proceed further (without loss of generality), let us perform a coordinate transformation on the gathered data. Define a unitary matrix $\boldsymbol{Q} = \boldsymbol{Q}_1 \otimes \boldsymbol{Q}_2$, where $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are chosen such that $\boldsymbol{Q}_1\boldsymbol{p}_1 = \boldsymbol{e}_1$ and $\boldsymbol{Q}_2\boldsymbol{p}_2 = \boldsymbol{e}_1$. Rotate the axes using $\boldsymbol{Q}$, yielding transformed data $\boldsymbol{x}_i = \boldsymbol{Q}\boldsymbol{x}_i^0$ and $\boldsymbol{d}_i = \boldsymbol{Q}\boldsymbol{d}_i^0$, for $i = 1, \ldots, H+K$. Then, Problem (1) can be equivalently written in a canonical form

$$\begin{cases} \mathrm{H}_1 : \boldsymbol{x}_i = \alpha_i \boldsymbol{e}_1 + \boldsymbol{d}_i, \\ \mathrm{H}_0 : \boldsymbol{x}_i = \boldsymbol{d}_i, \end{cases} \quad i = 1, \ldots, H. \tag{2}$$

Note that $\{\boldsymbol{d}_i\}_{i=1}^{H+K} \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(\boldsymbol{0}, \boldsymbol{M}_1 \otimes \boldsymbol{M}_2)$, where $\boldsymbol{M}_j \triangleq \boldsymbol{Q}_j \boldsymbol{M}_j^0 \boldsymbol{Q}_j^{\mathsf{H}}$, $j = 1, 2$.

To simplify notations, let us define the primary and secondary data matrices respectively as

$$\boldsymbol{X}_{\mathcal{H}} = \left[\boldsymbol{x}_1 \cdots \boldsymbol{x}_H\right], \quad \boldsymbol{X}_{\mathcal{K}} = \left[\boldsymbol{x}_{H+1} \cdots \boldsymbol{x}_{H+K}\right], \tag{3}$$

and their corresponding index sets as $\mathcal{H} = \{1, \ldots, H\}$ and $\mathcal{K} = \{H+1, \ldots, H+K\}$. Let $\boldsymbol{X}_{\text{all}} \in \mathbb{C}^{N\times(H+K)}$ denote the entire data matrix, i.e., $\boldsymbol{X}_{\text{all}} = [\boldsymbol{X}_{\mathcal{H}}, \boldsymbol{X}_{\mathcal{K}}]$. Further, define

$$\boldsymbol{X}_i = \mathrm{unvec}_{N_2,N_1}(\boldsymbol{x}_i), \; i = 1, \ldots, H+K, \tag{4}$$

where $\mathrm{unvec}_{N_2,N_1}(\cdot)$ reshapes a vector of size N into an $N_2 \times N_1$ matrix, serving as an inverse operation of the vec stacking.

III. Group Invariance Study

A key challenge in the hypothesis testing problem (2) lies in handling the nuisance parameters $(\boldsymbol{M}_1, \boldsymbol{M}_2)$ and achieving the desirable CFAR property. The *principle of invariance* [11], [19] addresses this challenge by focusing on decision rules enjoying some symmetry (or mathematically, invariance) under a specific group. This symmetry facilitates a reduction on the dimensionality of the parametric space, thereby potentially ensuring the CFAR property.

Leveraging this principle, we first identify an appropriate transformation group that preserves the structure of Problem (2) and effectively compresses the null parameter space into a single orbit. Subsequently, an MI, which plays a fundamental role in group invariance analysis as any invariant test must functionally depend on it, is obtained. To further characterize invariant tests, an induced MI is established, precisely representing the reduced parameter space upon which the distribution of any invariant test depends. It allows to verify that the invariance implies CFARity.

A. Transformation Group

Let $\mathscr{L}_n$ denote the set of all $n \times n$ invertible upper triangular matrices with positive real diagonal entries, and $\mathscr{U}_n$ the group of n-dimensional unitary matrices. Define the direct product of groups as $\mathscr{T} \triangleq \mathscr{L}_{N_1} \times \mathscr{L}_{N_2} \times \mathscr{U}_{H,K}$, where

$$\mathscr{U}_{H,K} = \left\{ \begin{bmatrix} \boldsymbol{U}_H & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_K \end{bmatrix} \middle| \boldsymbol{U}_H \in \mathscr{U}_H, \boldsymbol{U}_K \in \mathscr{U}_K \right\}. \tag{5}$$

Note that $\mathscr{T}$ also forms a group under the composition operation $\circ$ defined as $(\boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{U}_{H,K}) \circ (\boldsymbol{L}_1', \boldsymbol{L}_2', \boldsymbol{U}_{H,K}') = (\boldsymbol{L}_1'\boldsymbol{L}_1, \boldsymbol{L}_2'\boldsymbol{L}_2, \boldsymbol{U}_{H,K}\boldsymbol{U}_{H,K}')$. This group leaves the hypothesis test problem (2) invariant under the action

$$t(\boldsymbol{X}_{\text{all}}) = (\boldsymbol{L}_1 \otimes \boldsymbol{L}_2)\boldsymbol{X}_{\text{all}}\boldsymbol{U}, \; \forall t = (\boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{U}_{H,K}) \in \mathscr{T}. \tag{6}$$

as it can be readily verified that these transformations preserve the Gaussian distribution, the Kronecker covariance structure, and the partition of the parameter space.

B. Maximal Invariant and Induced Maximal Invariant

This subsection derives the MI with respect to $\mathscr{T}$ and the corresponding induced MI in the parameter space. It is worth noting that MIs are not unique, but all MIs are equivalent.

To obtain a meaningful MI, we begin by examining a commonly used estimator for the Kronecker-structured covariance matrix: the Kronecker MLE. Let $\mathcal{I} \subseteq \mathcal{H} \cup \mathcal{K}$. Based on a subset of observations $\{\boldsymbol{x}_i\}_{i\in\mathcal{I}}$ that are assumed to be free of target signal, the optimal first-order conditions suggest that the Kronecker MLE of $(\boldsymbol{M}_1, \boldsymbol{M}_2)$, if it exists, satisfies the following equations [20], [21]

$$\begin{cases} \boldsymbol{M}_1 = \dfrac{1}{N_2 I} \displaystyle\sum_{i\in\mathcal{I}} \boldsymbol{X}_i^{\mathsf{T}} \boldsymbol{M}_2^{-\mathsf{T}} \boldsymbol{X}_i^*, \\ \boldsymbol{M}_2 = \dfrac{1}{N_1 I} \displaystyle\sum_{i\in\mathcal{I}} \boldsymbol{X}_i \boldsymbol{M}_1^{-\mathsf{T}} \boldsymbol{X}_i^{\mathsf{H}}, \end{cases} \tag{7}$$

Notations—Throughout this paper, scalars are denoted by regular letters, vectors by boldface lowercase letters, and matrices by boldface uppercase letters. $\mathbb{C}^{M\times N}$ is the set of $M \times N$ complex matrices. $\mathbb{H}_{++}^N$ is the set of $N \times N$ positive definite Hermitian matrices. Symbols $(\cdot)^{\mathsf{T}}$, $(\cdot)^*$, and $(\cdot)^{\mathsf{H}}$ represent transpose, complex conjugate, and conjugate transpose, respectively. $\boldsymbol{0}$ denotes a null vector or matrix of appropriate size. $\boldsymbol{e}_1$ is an elementary vector of appropriate dimension, with 1 at the first entry and 0 otherwise. The Euclidean norm of a vector $\boldsymbol{a}$ is written as $\|\boldsymbol{a}\|$. Matrix operations such as determinant, exponential of the trace, Frobenius norm, and vectorization of a matrix $\boldsymbol{A}$ are denoted by $|\boldsymbol{A}|$, $\mathrm{etr}(\boldsymbol{A})$, $\|\boldsymbol{A}\|_{\mathrm{F}}$, and $\mathrm{vec}(\boldsymbol{A})$, respectively. For a matrix $\mathbf{A} \in \mathbb{H}_{++}^n$, $\mathrm{chol}(\mathbf{A})$ denotes its unique upper Cholesky factor. For a set $\mathcal{A}$, $|\mathcal{A}|$ denotes its cardinality. $\lfloor x \rfloor$ is the floor function for the real number x. The symbol $\sim$ denotes "distributed as," and $\overset{\text{i.i.d.}}{\sim}$ means "independent and identically distributed (i.i.d.)." $\boldsymbol{x} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ means that $\boldsymbol{x}$ is a complex circularly symmetric Gaussian vector with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$.

where $\boldsymbol{X}_i \in \mathbb{C}^{N_2 \times N_1}$ is defined in (4), and $I \triangleq |\mathcal{I}| \geq \lfloor N_1/N_2 + N_2/N_1 \rfloor + 1$ so as to guarantee the existence and uniqueness of the Kronecker MLE [22], [23] . The solution to (7) can be efficiently computed using the so-called "flip-flop" algorithm [20], [21], [24]. The Kronecker MLE obtained from $\{\boldsymbol{x}_i\}_{i \in \mathcal{I}}$ is denoted as $(\widehat{\boldsymbol{M}}_{1,\mathcal{I}}, \widehat{\boldsymbol{M}}_{2,\mathcal{I}})$. Specifically, we can take $\mathcal{I} = \mathcal{K}$, or $\mathcal{I} = \mathcal{H} \cup \mathcal{K}$ under H_0.

Proposition 1: An MI with respect to $\mathscr{T}$ is given by $\{\boldsymbol{S}_{\mathcal{H}}(\boldsymbol{X}_{\text{all}}), \boldsymbol{S}_{\mathcal{K}}(\boldsymbol{X}_{\text{all}})\}$, where

$$\boldsymbol{S}_{\mathcal{H}}(\boldsymbol{X}_{\text{all}}) = \widehat{\boldsymbol{M}}^{-1/2} \boldsymbol{X}_{\mathcal{H}} \boldsymbol{X}_{\mathcal{H}}^{\mathsf{H}} (\widehat{\boldsymbol{M}}^{-1/2})^{\mathsf{H}}, \tag{8}$$

$$\boldsymbol{S}_{\mathcal{K}}(\boldsymbol{X}_{\text{all}}) = \widehat{\boldsymbol{M}}^{-1/2} \boldsymbol{X}_{\mathcal{K}} \boldsymbol{X}_{\mathcal{K}}^{\mathsf{H}} (\widehat{\boldsymbol{M}}^{-1/2})^{\mathsf{H}}, \tag{9}$$

with $\widehat{\boldsymbol{M}}^{-1/2} = \operatorname{chol}(\widehat{\boldsymbol{M}}_{1,\mathcal{K}}^{-1}) \otimes \operatorname{chol}(\widehat{\boldsymbol{M}}_{2,\mathcal{K}}^{-1})$.

Invariance not only reduces the dataset to an MI statistic but also shrinks the parameter space. Let $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_H] \in \mathbb{C}^{1 \times H}$. The transformation group $\mathscr{T}$ induces the following action on the parameter space:

$$\bar{t}(\boldsymbol{\alpha}, \boldsymbol{M}_1, \boldsymbol{M}_2) = \Big\{ \ell_{1,11} \ell_{2,11} \boldsymbol{\alpha} \boldsymbol{U}_H, \boldsymbol{L}_1 \boldsymbol{M}_1 \boldsymbol{L}_1^{\mathsf{H}}, \boldsymbol{L}_2 \boldsymbol{M}_2 \boldsymbol{L}_2^{\mathsf{H}} \Big\},$$

where $\ell_{i,11}$ denotes the $(1,1)$-th entry of $\boldsymbol{L}_i$, for $i = 1, 2$, and $\boldsymbol{U}_H$ is the upper left block of $\boldsymbol{U} \in \mathscr{U}_{H,K}$. The set of all such induced transformations constitutes a group, denoted as $\overline{\mathscr{T}}$.

Proposition 2: An induced MI, i.e., an MI with respect to the induced group $\overline{\mathscr{T}}$ acting on the parameter space $(\boldsymbol{\alpha}, \boldsymbol{M}_1, \boldsymbol{M}_2)$, is given by

$$s_{\text{parm}} = \frac{\|\boldsymbol{\alpha}\|^2}{m_{1,1\cdot 2} m_{2,1\cdot 2}}, \tag{10}$$

where $m_{j,1\cdot 2}$ denotes the Schur complement of the (2, 2)-block in $\boldsymbol{M}_j$ according to the following partition

$$\boldsymbol{M}_j = \begin{bmatrix} m_{j,11} & \boldsymbol{m}_{j,12} \\ \boldsymbol{m}_{j,12}^{\mathsf{H}} & \boldsymbol{M}_{j,22} \end{bmatrix}, \quad j = 1, 2;$$

i.e., $m_{j,1\cdot 2} = m_{j,11} - \boldsymbol{m}_{j,12} \boldsymbol{M}_{j,22}^{-1} \boldsymbol{m}_{j,12}^{\mathsf{H}}$, for $j = 1, 2$.

Note that the induced MI s_{parm} reduces to zero under H_0, thereby ensuring the CFAR property for each invariant detector.

IV. Detector Design

This section focuses on the design of adaptive detectors for Problem (2). The multiple (intrinsic) dimensions of the maximal invariant suggest that in general no uniformly most powerful invariant (UMPI) test exists for this problem[1].

The invariant detection framework developed in Section III provides a clear route to CFAR detector design: it is possible to design detection strategies according to well-established criteria (such as GLRT, Rao test, and Wald test) which are invariant under transformations that leave the hypothesis testing problem itself invariant (possibly under some mild technical conditions) [3], [26]. Moreover, the CFARity of other devised detectors (e.g., according to the well-known two-step design paradigm) can be claimed by expressing them via MI.

[1]In this respect, note that as $H = 1$ and $N_2 = 1$, Problem (2) boils down to the unstructured case, where the UMPI does not exist [25].

A. Two-Step Adaptive Detector

When the covariance matrix is perfectly known, detectors established based on the GLRT, Rao test, and Wald test criteria have been derived in [27]–[29]; remarkably, under this ideal condition, all three detectors coincide. According to the widely adopted two-step strategy, the detectors become adaptive by replacing the unknown covariance $\boldsymbol{M}_1 \otimes \boldsymbol{M}_2$ with its Kronecker MLE, $(\widehat{\boldsymbol{M}}_{1,\mathcal{K}}, \widehat{\boldsymbol{M}}_{2,\mathcal{K}})$, computed from the secondary data.

Specifically, substituting the MLE $(\widehat{\boldsymbol{M}}_{1,\mathcal{K}} \otimes \widehat{\boldsymbol{M}}_{2,\mathcal{K}})$ into the ideal architectures yields the following unified expression for the two-step GLRT / Wald test / Rao test statistics:

$$\begin{aligned} &\frac{\boldsymbol{e}_1^{\mathsf{T}} (\widehat{\boldsymbol{M}}_{1,\mathcal{K}} \otimes \widehat{\boldsymbol{M}}_{2,\mathcal{K}})^{-1} \boldsymbol{X}_{\mathcal{H}} \boldsymbol{X}_{\mathcal{H}}^{\mathsf{H}} (\widehat{\boldsymbol{M}}_{1,\mathcal{K}} \otimes \widehat{\boldsymbol{M}}_{2,\mathcal{K}})^{-1} \boldsymbol{e}_1}{\boldsymbol{e}_1^{\mathsf{T}} (\widehat{\boldsymbol{M}}_{1,\mathcal{K}} \otimes \widehat{\boldsymbol{M}}_{2,\mathcal{K}})^{-1} \boldsymbol{e}_1} \\ &= \boldsymbol{e}_1^{\mathsf{T}} \boldsymbol{S}_{\mathcal{H}}(\boldsymbol{X}_{\text{all}}) \boldsymbol{e}_1, \end{aligned} \tag{11}$$

which is a function of the MI component $\boldsymbol{S}_{\mathcal{H}}$. Consequently, by the invariance analysis of Section III, this two-step adaptive detector possesses the CFAR property.

B. One-Step GLRT

This subsection considers the one-step GLRT for (2), which is formulated according to

$$T_{\text{GLRT}}(\boldsymbol{X}_{\text{all}}) = \frac{\max\limits_{\boldsymbol{\alpha}, \boldsymbol{M}_1, \boldsymbol{M}_2} f(\boldsymbol{X}_{\text{all}}; \boldsymbol{\alpha}, \boldsymbol{M}_1, \boldsymbol{M}_2 \mid \mathrm{H}_1)}{\max\limits_{\boldsymbol{M}_1, \boldsymbol{M}_2} f(\boldsymbol{X}_{\text{all}}; \boldsymbol{M}_1, \boldsymbol{M}_2 \mid \mathrm{H}_0)}, \tag{12}$$

where, for $b \in \{0, 1\}$,

$$\begin{aligned} &f(\boldsymbol{X}_{\text{all}} \mid \mathrm{H}_b) = \\ &\Big[\pi^N |\boldsymbol{M}_1 \otimes \boldsymbol{M}_2|\Big]^{-(H+K)} \operatorname{etr}\Big\{ -(\boldsymbol{M}_1 \otimes \boldsymbol{M}_2)^{-1} \\ &\Big[\sum_{h \in \mathcal{H}} (\boldsymbol{x}_h - b\alpha_h \boldsymbol{e}_1)(\boldsymbol{x}_h - b\alpha_h \boldsymbol{e}_1)^{\mathsf{H}} + \sum_{k \in \mathcal{K}} \boldsymbol{x}_k \boldsymbol{x}_k^{\mathsf{H}}\Big]\Big\}. \end{aligned}$$

Evidently, the Kronecker MLE given by (7) with $\mathcal{I} = \mathcal{H} \cup \mathcal{K}$ is the solution to the optimization of the likelihood function under H_0. As a result,

$$\max_{\boldsymbol{M}_1, \boldsymbol{M}_2} f(\boldsymbol{X}_{\text{all}} | \mathrm{H}_0) \propto \big|\widehat{\boldsymbol{M}}_{1,\mathcal{H}\cup\mathcal{K}} \otimes \widehat{\boldsymbol{M}}_{2,\mathcal{H}\cup\mathcal{K}}\big|^{-(H+K)}. \tag{13}$$

Under H_1, the optimization can proceed via an alternating maximization scheme:

$$\begin{aligned} \boldsymbol{\alpha}^{(t)} &= \arg\max_{\boldsymbol{\alpha}} f(\boldsymbol{X}_{\text{all}}; \boldsymbol{\alpha}, \boldsymbol{M}_1^{(t-1)}, \boldsymbol{M}_2^{(t-1)} \mid \mathrm{H}_1) \\ &= \frac{\boldsymbol{e}_1^{\mathsf{T}} (\boldsymbol{M}_1^{(t-1)} \otimes \boldsymbol{M}_2^{(t-1)})^{-\mathsf{T}} \boldsymbol{X}_{\mathcal{H}}}{\boldsymbol{e}_1^{\mathsf{T}} (\boldsymbol{M}_1^{(t-1)} \otimes \boldsymbol{M}_2^{(t-1)})^{-1} \boldsymbol{e}_1}, \end{aligned} \tag{14}$$

and

$$(\boldsymbol{M}_1^{(t)}, \boldsymbol{M}_2^{(t)}) = \arg\max_{(\boldsymbol{M}_1, \boldsymbol{M}_2)} f(\boldsymbol{X}_{\text{all}}; \boldsymbol{\alpha}^{(t)}, \boldsymbol{M}_1, \boldsymbol{M}_2 \mid \mathrm{H}_1), \tag{15}$$

By the first-order condition, (15) can be solved using a similar flip-flop algorithm as for (7) with the dataset $\{\boldsymbol{x}_h - \alpha_h^{(t)} \boldsymbol{e}_1, \boldsymbol{x}_k\}_{h \in \mathcal{H}}^{k \in \mathcal{K}}$. Algorithm 1 summarizes the procedure to compute the MLE under H_1, which demonstrates promising

Algorithm 1: Algorithm for the MLE under H_1

Input: $\{\boldsymbol{x}_h, \boldsymbol{x}_k\}$, tolerance δ, maximum iterations n_{it}.

Output: $\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{M}}_{1,\mathrm{H}_1}, \widehat{\boldsymbol{M}}_{2,\mathrm{H}_1}$.

Initiate $\boldsymbol{M}_1^{(0)} = \widehat{\boldsymbol{M}}_{1,\mathcal{K}}, \boldsymbol{M}_2^{(0)} = \widehat{\boldsymbol{M}}_{2,\mathcal{K}}$, $t = 0$;

repeat

Compute $\boldsymbol{\alpha}^{(t)}$ using (14);

Set $\tilde{\boldsymbol{X}}_h = \operatorname{unvec}_{N_2,N_1}(\boldsymbol{x}_h - \alpha_h^{(t)})$, for $h \in \mathcal{H}$;

Set $\tilde{\boldsymbol{X}}_k = \boldsymbol{X}_k$, for $k \in \mathcal{K}$;

Compute $(\boldsymbol{M}_1^{(t)}, \boldsymbol{M}_2^{(t)})$ by solving the following equations using the "flip-flop" algorithm (initiated with $(\boldsymbol{M}_1^{(t-1)}, \boldsymbol{M}_2^{(t-1)})$):

$$\begin{cases} \boldsymbol{M}_1 = \dfrac{1}{N_2(H+K)} \displaystyle\sum_{i \in \mathcal{H}\cup\mathcal{K}} \tilde{\boldsymbol{X}}_i^{\mathsf{T}} \boldsymbol{M}_2^{-\mathsf{T}} \tilde{\boldsymbol{X}}_i^*, \\ \boldsymbol{M}_2 = \dfrac{1}{N_1(H+K)} \displaystyle\sum_{i \in \mathcal{H}\cup\mathcal{K}} \tilde{\boldsymbol{X}}_i \boldsymbol{M}_1^{-\mathsf{T}} \tilde{\boldsymbol{X}}_i^{\mathsf{H}}, \end{cases};$$

$t \leftarrow t + 1$;

until $\|\boldsymbol{M}_1^{(t)} - \boldsymbol{M}_1^{(t-1)}\|_{\mathrm{F}} + \|\boldsymbol{M}_2^{(t)} - \boldsymbol{M}_2^{(t-1)}\|_{\mathrm{F}} \leq \delta$ *or* $t > n_{\text{it}}$;

return $\widehat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}^{(t)}, \widehat{\boldsymbol{M}}_{1,\mathrm{H}_1} = \boldsymbol{M}_1^{(t)}, \widehat{\boldsymbol{M}}_{2,\mathrm{H}_1} = \boldsymbol{M}_2^{(t)}$.

empirical performance in our numerical examples under the sample size constraint $H + K \geq \lfloor N_1/N_2 + N_2/N_1 \rfloor + 1$. The maximized likelihood function under H_1 is given by

$$\max_{\boldsymbol{\alpha},\boldsymbol{M}_1,\boldsymbol{M}_2} f(\boldsymbol{X}_{\text{all}}|\mathrm{H}_1) \propto \left|\widehat{\boldsymbol{M}}_{1,\mathrm{H}_1} \otimes \widehat{\boldsymbol{M}}_{2,\mathrm{H}_1}\right|^{-(H+K)}. \quad (16)$$

Then, the one-step GLRT given by (12) has the following expression (neglecting some irrelevant constants):

$$T_{\text{GLRT}}(\boldsymbol{X}_{\text{all}}) \propto \left(\frac{\left|\widehat{\boldsymbol{M}}_{1,\mathcal{H}\cup\mathcal{K}} \otimes \widehat{\boldsymbol{M}}_{2,\mathcal{H}\cup\mathcal{K}}\right|}{\left|\widehat{\boldsymbol{M}}_{1,\mathrm{H}_1} \otimes \widehat{\boldsymbol{M}}_{2,\mathrm{H}_1}\right|} \right)^{(H+K)}. \quad (17)$$

Notably, the one-step GLRT indeed corresponds to the standard GLRT for Problem 2, and thus, by [26], it is an invariant test with respect to $\mathscr{T}$. Consequently, it achieves the CFAR property (assuming convergence of the algorithm and uniqueness of the solutions) by the results established in Section III.

V. Numerical Examples

This section presents numerical results to assess the performance of the proposed detectors. Consider a pulsed Doppler radar with $N_2 = 5$ uniformly spaced array elements, and let the number of pulses per coherent processing interval be $N_1 = 4$. The spatial steering vector at spatial frequency f_s is given by $\boldsymbol{p}_2(f_s) = \frac{1}{\sqrt{N_2}}[1, \; e^{\imath 2\pi f_s}, \ldots, e^{\imath 2\pi (N_2-1) f_s}]^{\mathsf{T}}$, and the temporal steering vector at Doppler frequency f_d is $\boldsymbol{p}_1(f_d) = \frac{1}{\sqrt{N_1}}[1, \; e^{\imath 2\pi f_d}, \ldots, e^{\imath 2\pi (N_1-1) f_d}]^{\mathsf{T}}$. The covariance matrix of the disturbance $\boldsymbol{d}_i^0$ is $\sigma_c^2(\boldsymbol{M}_1^0 \otimes \boldsymbol{M}_2^0)$, where $\boldsymbol{M}_1^0$ and $\boldsymbol{M}_2^0$ are Toeplitz matrices with entries $\rho^{|m-l|}$ and $\rho^{|m-l|^2} e^{-\imath 2\pi \cdot 0.05(m-l)}$, respectively, and σ_c^2 is adjusted to achieve a desired clutter-to-noise ratio (CNR).

Fixing $H = 4$ and setting the nominal false alarm rate to $P_f = 10^{-4}$, we consider the following typical sample sizes of the secondary data:

- No secondary training data case: $K = 0$;
- Sample-starved case: $K = 3$;
- Moderate sample size case: $K = 30$.

The abbreviations of the considered detectors and their implementation condition are listed below:

- KP-GLRT2S: the two-step test given in (11), applicable for $K \geq \lfloor N_1/N_2 + N_2/N_1 \rfloor + 1$;
- KP-GLRT1S: the one-step GLRT given in (17), applicable for $H + K \geq \lfloor N_1/N_2 + N_2/N_1 \rfloor + 1$;
- U-GLRT1S, U-GLRT2S, and U-GASD: classical unstructured detectors proposed in [27], [28], applicable only for $K \geq N_1 N_2$;
- Clairvoyant benchmark: the ideal two-step GLRT with perfect knowledge of $\boldsymbol{M}_1 \otimes \boldsymbol{M}_2$.

Fig. 1 shows the empirical false alarm rates (P_f) for $K = 3$, demonstrating the CFAR behavior of the proposed detectors in response to variations in the correlation parameter ρ. This also validates the theoretical CFAR analysis in Section IV.

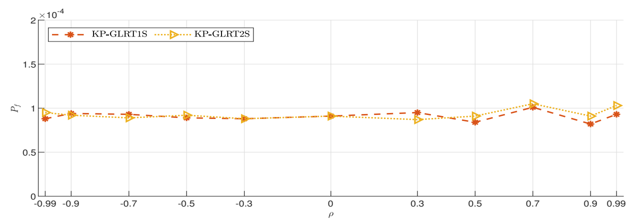


Fig. 1. Actual P_f curves with the nominal false alarm rate 10^{-4}, $K = 3$, $f_d = 0.1$, and $\psi = 0$.

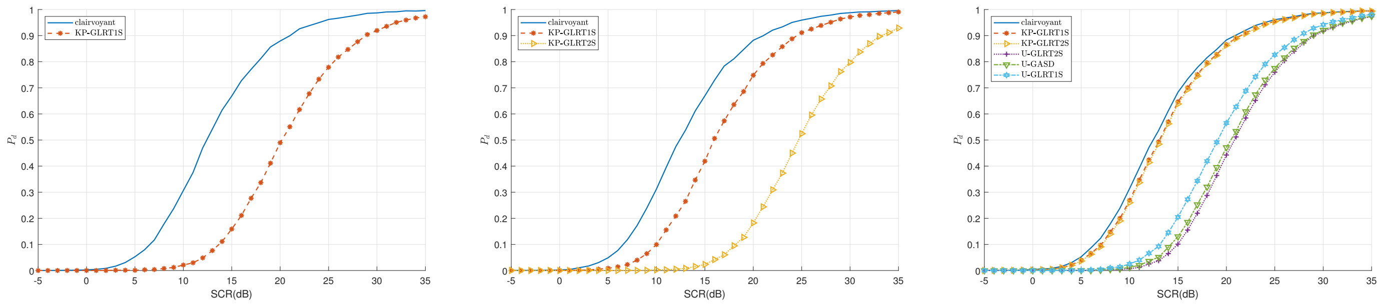
Furthermore, we evaluate the detection probabilities of the considered detectors versus the Signal-to-Clutter Ratio (SCR) under different training data sizes. The SCR is defined as

$$\mathrm{SCR} = \mathbb{E}(\|\boldsymbol{\alpha}\|^2)\left(\boldsymbol{p}_1 \otimes \boldsymbol{p}_2\right)^{\mathsf{H}}\left[\sigma_c^2 \boldsymbol{M}_1^0 \otimes \boldsymbol{M}_2^0\right]^{-1}\left(\boldsymbol{p}_1 \otimes \boldsymbol{p}_2\right),$$

where the target amplitude vector is simulated as $\boldsymbol{\alpha} = \frac{a_0}{\sqrt{H}}[1, \ldots, 1]$, with $a_0 \sim \mathcal{CN}(0, \sigma_s^2)$. Fig. 2(a) considers the case $K = 0$ (i.e., no secondary training data), where only the KP-GLRT1S can operate effectively, and it displays satisfactory performance in the plot. Fig. 2(b) ($K = 3$) shows that the KP-GLRT1S significantly outperforms the KP-GLRT2S, highlighting the benefit of jointly exploiting the primary and secondary data. Fig. 2(c) ($K = 30$) compares the proposed Kronecker-structured detectors with conventional detectors based on unstructured covariance matrix estimators. The KP-GLRT1S and the KP-GLRT2S show comparable performance in the moderate sample size case. These unstructured competitions exhibit a performance gap exceeding 6 dB compared to our proposed detectors, which effectively leverage the Kronecker structure, thus requiring fewer samples to achieve a high detection probability.

VI. Conclusions

In this paper, we have explored the detection of range-spread targets in Gaussian disturbance with Kronecker structured covariance matrix. After transforming the original hypotheses testing problem into its canonical form, we identified a



(a) $K = 0$ (b) $K = 3$ (c) $K = 30$

Fig. 2. P_d versus SCR with $H = 4$, $\sigma_c^2 = 30$ dB, $\rho = 0.9$, $f_d = 0.1$, and $\psi = 0$.

transformation group $\mathscr{T}$ whose action preserves the problem structure. This invariance effectively compresses the nuisance parameters, ensuring that invariance under $\mathscr{T}$ implies the CFAR property. With respect to $\mathscr{T}$, an MI and an induced MI have been provided. Subsequently, we derived the two-step adaptive detector which incorporates the Kronecker MLE from the secondary data, as well as the one-step GLRT that applies the GLRT test criterion directly to the entire dataset. Both the detection statistics are invariant under $\mathscr{T}$, as verified by one being expressed as a function of the MI and the other by an invariance property of the standard GLRT test criterion. Thus, their CFAR properties are ensured according to the invariant detection framework established in this paper. Numerical results demonstrate that these detectors maintain robust CFAR behavior and confirm the performance gains offered by exploitation of the Kronecker covariance structure, particularly when the training data set is limited.

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