Adaptive Detection of a Gaussian Signal Known Only to Lie in an Unknown Subspace of Known Dimension

1st Pia Addabbo
Università degli Studi "Giustino Fortunato"
Benevento, Italy
p.addabbo@unifortunato.eu

2nd Danilo Orlando
Dipartimento di Ingegneria dell'Informazione
Università di Pisa
Pisa, Italy
danilo.orlando@unipi.it

3rd Giuseppe Ricci
Dipartimento di Ingegneria dell'Innovazione
Università del Salento
Lecce, Italy
giuseppe.ricci@unisalento.it

4th Louis L. Scharf
Departments of Mathematics and Statistics
Colorado State University
Fort Collins, CO, USA
scharf@colostate.edu

***Abstract*—This paper addresses the problem of detecting multidimensional subspace signals in disturbance (interference plus noise) of unknown covariance. It is assumed that a primary channel of measurements, possibly consisting of signal plus disturbance, is augmented with a secondary channel of measurements containing only disturbance. The interference plus noise terms in these two channels share an unknown covariance matrix; as to the signal it belongs to a subspace known only by its dimension and consecutive visits to the subspace are constrained by a prior distribution. The performance of the related detector is extensively investigated in new scenarios that have not been taken into consideration so far. Specifically, these scenarios provide the radar engineer with further insights about the detector behavior and comprise a mismatch between the nominal (that is a design parameter) and actual (that depends on data) subspace dimension as well as the case where the radar system is under the attack of a noise-like jammer.**

***Index Terms*—Adaptive Detection, Subspace Model, Generalized Likelihood Ratio Test.**

I. Introduction

In real radar systems equipped with an array of sensors, the array mainbeam is steered by applying specific weights to each tile. However, very often, an uncertainty related to the array pointing direction may exist due to hardware, mutual coupling, calibration residuals, and so on [1]. The subspace paradigm arises from the need to account for this uncertainty and to prevent detection performance degradation due to the presence of mismatched signals [2]. The general problem of matched and adaptive subspace detection of point-like targets in Gaussian and non-Gaussian disturbance has been addressed by many authors, beginning with the seminal work of Kelly and Forsythe [3], [4]. The innovation of [3] was to introduce a *homogeneous* secondary channel of signal-free measurements whose unknown covariance matrix was equal to the unknown covariance matrix of primary (or test) measurements. Likelihood theory was then used to derive what is now called the Kelly detector. In [4], several important generalizations have been addressed including the case of multiple primary measurements. These papers were followed by other important contributions [5]–[14]. However, most of these works address adaptive detection in what might be called a first-order (signal) model for measurements. That is, the measurements under test may contain a signal in a known subspace embedded in Gaussian disturbance of unknown covariance, but no prior distribution is assigned to the location of the signal in the subspace.

The first attempt to replace this model by a second-order (signal) model was made in [15], where the authors used a Gaussian model for the signal. The covariance matrix for the signal was constrained by a known subspace model. In [15], a detector based on the estimate-and-plug (EP) strategy was proposed. The EP strategy is a two-step design approach: first the generalized likelihood ratio test (GLRT) is implemented based on primary measurements only, herein assuming a known covariance matrix up to a multiplicative factor. Then the detector is made fully adaptive by estimating the structure of the covariance matrix based on the secondary measurements.

The main goal of the current paper is the analysis of a decision scheme aimed at detecting a Gaussian signal (second-order) belonging to a subspace known only by its dimension, recently introduced in [16]. Specifically, this paper will extend the analysis of such a detector contained in [17] to the case where a mismatch exists between the nominal and the actual subspace dimension. In addition, the detection performance is investigated under the action of a cover jammer that affects both primary and secondary channels. Our results are motivated by the problem of detecting range-spread targets from an active radar system.

The remainder of this paper is organized as follows. The next section provides a formal statement of the second-order detection problem. Then, Section III briefly recalls the structure of the decision scheme. The analysis under mismatch and jamming is provided in Section IV, whereas Section V draws concluding remarks and describes future research tracks.

A. Notation

In the sequel, vectors and matrices are denoted by boldface lower-case and upper-case letters, respectively. Symbols $\det(\cdot)$, $\mathrm{Tr}\,(\cdot)$, $\mathrm{etr}\,\{\cdot\}$, $(\cdot)^T$, $(\cdot)^\dagger$, and $(\cdot)^{-1}$ denote the determinant, trace, exponential of the trace, transpose, conjugate transpose, and inverse, respectively. $\boldsymbol{A}^{1/2}$ will denote the square root of a Hermitian, positive semidefinite matrix $\boldsymbol{A}$ while $\boldsymbol{A}^{-1/2}$ is the inverse of the square root of a Hermitian, positive definite matrix $\boldsymbol{A}$. As to numerical sets, $\mathbb{C}$ is the set of complex numbers with $|\cdot|$ the absolute value of $z \in \mathbb{C}$, $\mathbb{C}^{N\times M}$ is the Euclidean space of $(N \times M)$-dimensional complex matrices, and $\mathbb{C}^N$ is the Euclidean space of N-dimensional complex vectors. $\boldsymbol{I}_n$ and $\boldsymbol{0}_{m,n}$ stand for the $n \times n$ identity matrix and the $m \times n$ null matrix. $\langle \boldsymbol{H} \rangle$ denotes the space spanned by the columns of the matrix $\boldsymbol{H} \in \mathbb{C}^{N\times r}$. Given $a_1, \dots, a_N \in \mathbb{C}$, $\mathrm{diag}\,(a_1, \dots, a_N) \in \mathbb{C}^{N\times N}$ indicates the diagonal matrix whose ith diagonal element is a_i. We write $\boldsymbol{z} \sim \mathcal{CN}_N(\boldsymbol{x}, \boldsymbol{\Sigma})$ to say that the N-dimensional random vector $\boldsymbol{z}$ is a complex normal random vector with mean vector $\boldsymbol{x}$ and covariance matrix $\boldsymbol{\Sigma}$. Moreover, $\boldsymbol{Z} = [\boldsymbol{z}_1 \cdots \boldsymbol{z}_K] \sim \mathcal{CN}_{NK}(\boldsymbol{X}, \boldsymbol{I}_K \otimes \boldsymbol{\Sigma})$, with $\otimes$ denoting Kronecker product and $\boldsymbol{X} = [\boldsymbol{x}_1 \cdots \boldsymbol{x}_K]$, means that $\boldsymbol{z}_k \sim \mathcal{CN}_N(\boldsymbol{x}_k, \boldsymbol{\Sigma})$ and the columns of $\boldsymbol{Z}$ are statistically independent. The acronyms PDF and IID stand for probability density function and independent and identically distributed, respectively. $\widehat{\boldsymbol{R}}_i$ will denote the maximum likelihood (ML) estimate of $\boldsymbol{R}$ under the H_i hypothesis, $i = 0, 1$ (symbols defined in Section II). The subscript is discarded in case it is unnecessary.

II. Problem Formulation

For subsequent developments, let us denote by $\boldsymbol{Z}_P = [\boldsymbol{z}_1 \cdots \boldsymbol{z}_{K_P}] \in \mathbb{C}^{N\times K_P}$ the matrix of the measurements in the primary channel and by $\boldsymbol{Z}_S = [\boldsymbol{z}_{K_P+1} \cdots \boldsymbol{z}_{K_P+K_S}] \in \mathbb{C}^{N\times K_S}$ the matrix of the measurements in the secondary channel. In a radar problem, the measurements are N-dimensional vectors of space-time samples: the radar system transmits a burst of N_p radio frequency (RF) pulses and the baseband representations of the RF signals collected at the N_a antenna elements are sampled to form range-gate samples for each pulse; it turns out that $N = N_a N_p$. If the signal presence is sought in a subset of K_P range gates, the primary channel consists of NK_P samples. The samples corresponding to any range gate are arranged in a column vector $\boldsymbol{z}_k \in \mathbb{C}^N$. The secondary channel consists of the outputs of K_S properly selected range gates [18]. Finally, let $\boldsymbol{Z} = [\boldsymbol{Z}_P \ \boldsymbol{Z}_S] \in \mathbb{C}^{N\times K}$ be the overall data matrix with $K = K_P + K_S$.

The adaptive detection problem we are interested in can be formulated as the following hypothesis testing problem

$$H_0 : \begin{cases} \boldsymbol{Z}_P \sim \mathcal{CN}_{NK_P}(\boldsymbol{0}_{N,K_P}, \boldsymbol{I}_{K_P} \otimes \boldsymbol{R}) \\ \boldsymbol{Z}_S \sim \mathcal{CN}_{NK_S}(\boldsymbol{0}_{N,K_S}, \boldsymbol{I}_{K_S} \otimes \boldsymbol{R}) \end{cases}$$

$$H_1 : \begin{cases} \boldsymbol{Z}_P \sim \mathcal{CN}_{NK_P}(\boldsymbol{0}_{N,K_P}, \boldsymbol{I}_{K_P} \otimes (\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^\dagger + \boldsymbol{R})) \\ \boldsymbol{Z}_S \sim \mathcal{CN}_{NK_S}(\boldsymbol{0}_{N,K_S}, \boldsymbol{I}_{K_S} \otimes \boldsymbol{R}) \end{cases} \tag{1}$$

where $\boldsymbol{H} \in \mathbb{C}^{N\times r}$ is an unknown full column rank matrix, with known rank r, $r \leq N$, $\boldsymbol{R}_{xx} \in \mathbb{C}^{r\times r}$ is an unknown positive semidefinite covariance matrix (whose rank is less than or equal to r), while $\boldsymbol{R} \in \mathbb{C}^{N\times N}$ is an unknown positive definite matrix. Moreover, we suppose that $K_S \geq N$.

The joint PDF of primary and secondary data is given by

$$f_1(\boldsymbol{Z}; \boldsymbol{R}, \boldsymbol{R}_{xx}, \boldsymbol{H}) = \frac{\mathrm{etr}\left\{-\boldsymbol{R}^{-1}\boldsymbol{Z}_S\boldsymbol{Z}_S^\dagger\right\}}{\pi^{NK}} \times \frac{\mathrm{etr}\left\{-\left(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^\dagger + \boldsymbol{R}\right)^{-1}\boldsymbol{Z}_P\boldsymbol{Z}_P^\dagger\right\}}{\det^{K_P}(\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^\dagger + \boldsymbol{R})\det^{K_S}(\boldsymbol{R})} \tag{2}$$

under H_1 and it is expressed as

$$f_0(\boldsymbol{Z}; \boldsymbol{R}) = \frac{\mathrm{etr}\left\{-\boldsymbol{R}^{-1}(\boldsymbol{Z}_P\boldsymbol{Z}_P^\dagger + \boldsymbol{Z}_S\boldsymbol{Z}_S^\dagger)\right\}}{\pi^{NK}\det^K(\boldsymbol{R})} \tag{3}$$

under H_0. To derive the GLRT we have to compute the compressed likelihoods under each hypothesis. This task is the object of the next section.

III. Detector design

Firstly, observe that the computation of the compressed likelihood under H_0 is a well-known result, see for instance [3].

As to the likelihood under H_1, it is convenient to observe that in eq. (2) the parameters $\boldsymbol{H}$ and $\boldsymbol{R}_{xx}$ are both unknown, so $\boldsymbol{H}\boldsymbol{R}_{xx}\boldsymbol{H}^\dagger$ may be replaced by the unknown covariance matrix $\tilde{\boldsymbol{R}}_{xx}$. Thus, the log-likelihood under H_1 can be written as

$$\begin{aligned} L_1(\boldsymbol{R}, \tilde{\boldsymbol{R}}_{xx}; \boldsymbol{Z}) &= -NK\log\pi \\ &- K_P \log\det(\tilde{\boldsymbol{R}}_{xx} + \boldsymbol{R}) - \mathrm{Tr}\left[\left(\tilde{\boldsymbol{R}}_{xx} + \boldsymbol{R}\right)^{-1}\boldsymbol{S}_P\right] \\ &- K_S \log\det(\boldsymbol{R}) - \mathrm{Tr}\left[\boldsymbol{R}^{-1}\boldsymbol{S}_S\right] \end{aligned} \tag{4}$$

where $\boldsymbol{S}_P = \boldsymbol{Z}_P\boldsymbol{Z}_P^\dagger$ and $\boldsymbol{S}_S = \boldsymbol{Z}_S\boldsymbol{Z}_S^\dagger$ (and the matrix $\boldsymbol{S}_S$ is positive definite since $K_S \geq N$). Notice also that the rank of the matrix $\tilde{\boldsymbol{R}}_{xx}$ is less than or equal to r (in fact, the rank of $\boldsymbol{H}\boldsymbol{R}_{xx}^{1/2}$ is less than or equal to r). The compressed likelihood necessary to obtain the GLRT is given by the following theorem. The focus is on the case $r \leq K_P \leq N$ although extension to $K_P < r$ is straightforward. For the proof of the theorem see [16].

Theorem 1: Let $r \leq K_P \leq N$. Denote by $\boldsymbol{\Gamma} = \mathrm{diag}\,(\gamma_1, \dots, \gamma_N) \in \mathbb{R}^{N\times N}$, $\gamma_1 \geq \ldots \geq \gamma_N \geq 0$, the diagonal matrix containing the eigenvalues of $\boldsymbol{S}_S^{-1/2}\boldsymbol{S}_P\boldsymbol{S}_S^{-1/2}$ and by

$\boldsymbol{V} \in \mathbb{C}^{N \times N}$ the unitary matrix of the corresponding eigenvectors. Finally, let $\boldsymbol{K} = \boldsymbol{S}_S^{1/2}\boldsymbol{V} \in \mathbb{C}^{N \times N}$. The maximum of the left-hand side of (4) can be rewritten as

$$L_1(\widehat{\boldsymbol{R}}_1, \widehat{\widehat{\boldsymbol{R}}}_{xx}; \boldsymbol{Z}) = -NK \log \pi - NK$$
$$- 2K \log |\det(\boldsymbol{K})| + \sum_{i=1}^{N} \left[K \log \frac{K}{\gamma_i + \widehat{\lambda}_i} + K_S \log \widehat{\lambda}_i \right]$$

with

$$\widehat{\lambda}_i = \begin{cases} \max\left(\frac{K_S \gamma_i}{K_P}, 1\right), & i = 1, \dots, r, \\ 1, & i = r+1, \dots, N. \end{cases}$$

Finally, the GLRT for homogeneous environment and unknown subspace $\langle \boldsymbol{H} \rangle$, referred to in the following as second-order unknown subspace in homogeneous environment (SO-US-HE) detector, is

$$L_1(\widehat{\boldsymbol{R}}_1, \widehat{\widehat{\boldsymbol{R}}}_{xx}; \boldsymbol{Z}) - L_0(\widehat{\boldsymbol{R}}_0; \boldsymbol{Z}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta \tag{5}$$

with $L_0(\widehat{\boldsymbol{R}}_0; \boldsymbol{Z})$ the compressed likelihood under H_0 and η the threshold to be set according to the desired probability of false alarm (P_{fa}).

IV. Illustrative Examples and Discussion

In this section, Monte Carlo (MC) counting techniques are used to evaluate the performance of the SO-US-HE and this is compared to the performance of the corresponding EP approach proposed in [17].

The probability of detection (P_d) and the thresholds to guarantee a given P_{fa} are estimated over 10^3 and $100/P_{fa}$ independent MC trials, respectively. In all the illustrative examples we assume $N = 16$ and $P_{fa} = 10^{-3}$, $r = 2, 4$, $K_P = N$, and $K_S = 1.5 \cdot N$. The covariance matrix $\boldsymbol{R}$ of the disturbance is

$$\boldsymbol{R} = \boldsymbol{I}_N + \sigma_c^2 \boldsymbol{M}_c \tag{6}$$

with σ_c^2 accounting for a clutter-to-noise ratio of 30 dB assuming unit noise power. The (i, j)th entry of the clutter component $\boldsymbol{M}_c$ is $\rho_c^{|i-j|}$ with $\rho_c = 0.95$.

In the simulated scenario, the signal component in the ith vector $\boldsymbol{z}_i$, $i = 1, \dots, K_P$, is given by $\alpha_i \mathbf{v}(\phi_i)$, with

$$\mathbf{v}(\phi_i) = \frac{1}{\sqrt{N}} \left[1 \ e^{j\phi_i} \ \cdots \ e^{j(N-1)\phi_i} \right]^T; \tag{7}$$

the electrical angles ϕ_i are independent random variables uniformly distributed on $\Phi = [-\pi\beta, \pi\beta]$, where $\beta = \sin\theta$ with θ depending on the desired value of the rank r. The interval Φ is discretized using a step of 0.02 radians. As to the α_is, they are IID complex normal random variables, $\alpha_i \sim \mathcal{CN}_1(0, \sigma_\alpha^2)$, $i = 1, \dots, K_P$. More precisely, we estimate the actual rank r of the signal subspace by computing the matrix $\boldsymbol{R}_\beta \in \mathbb{C}^{N \times N}$, whose (m, n)th entry is given by [1]

$$\boldsymbol{R}_\beta(m, n) = 2\beta\pi \mathrm{sinc}((n - m)\beta)$$

and determining the corresponding number of eigenvalues significantly different from zero. In particular, we choose $\theta = 2\pi(3/360)$ and $\theta = 2\pi(8/360)$ radians to obtain $r = 2$ and $r = 4$.

The performance analysis considers different scenarios:

- matched case: the assumed signal subspace dimension, say $\tilde{r}$, coincides with the actual one r;
- mismatched case: the assumed subspace dimension differs from the actual one ($r > \tilde{r}$ or $r < \tilde{r}$);
- jamming scenario: the system is subjected to noise-like jamming (NLJ) with jammer-to-noise-ratio (JNR) given by JNR $= \sigma_j^2 = 30$ dB.

The results, presented in Figures 1-3, plot P_d vs the signal-to-interference-plus-noise ratio (SINR), defined as

$$\text{SINR} = \sigma_\alpha^2 \text{Tr}\left(\boldsymbol{V}_P^\dagger \boldsymbol{R}^{-1} \boldsymbol{V}_P \right)$$

with, in turn, $\boldsymbol{V}_P = [\mathbf{v}(\phi_1) \cdots \mathbf{v}(\phi_{K_P})]$. The figures provide insight into the robustness of the proposed detector under the different scenarios, comparing the performance of GLRT-based detectors and their EP counterparts.

Particularly, the top plot of Figure 1 corresponds to the matched case, (i.e., $r = \tilde{r} = 2$). When the subspace dimension is correct, the curves highlight that at 20 dB the P_d of the SO-US-HE is 0.9 while that of the EP-SO-US-HE is 0.8. Generally speaking, the EP-SO-US-HE experiences a slight performance degradation compared to the SO-US-HE. The bottom plot illustrates the mismatched scenario, where the assumed subspace dimension is greater than the actual one ($r = 2, \tilde{r} = 4$). It turns out that both detectors exhibit a small gain with respect to the perfect matched case particularly at intermediate SINR values. This behavior can be explained by observing the decision statistics of the two detectors. In fact, increasing $\tilde{r}$ might include directions that improve the collected energy.

Figure 2 shows a performance degradation in the mismatched scenario where $r > \tilde{r}$. Specifically, in the matched case ($r = \tilde{r} = 4$), both the SO-US-HE and the EP-SO-US-HE achieve high P_d values at lower SINR compared to Figure 1. Moreover, in the mismatched case ($r = 4, \tilde{r} = 2$), the detection performance is significantly worse compared to the matched case. As a matter of fact, reducing the assumed subspace dimension ($\tilde{r} < r$) prevents the detector from capturing the full signal structure (and, hence, energy) impairing the detection performance.

Figure 3 investigates the impact of NLJ on the detection performance, comparing $r = 2$ and $r = 4$ both under matched conditions. The presence of the NLJ alters the overall disturbance covariance matrix as follows

$$\boldsymbol{R} = \boldsymbol{I}_N + \sigma_c^2 \boldsymbol{M}_c + \sigma_j^2 \mathbf{v}(\phi_j)\mathbf{v}(\phi_j)^\dagger, \tag{8}$$

where we recall that $\sigma_j^2 > 0$ is the jammer power set according to the JNR and $\phi_j = \pi \sin(2\pi/20)$ is the jammer electrical angle. From a comparison with the previous matched case without the action of the NLJ, it turns out that the detection performance is approximately invariant for both values of r.

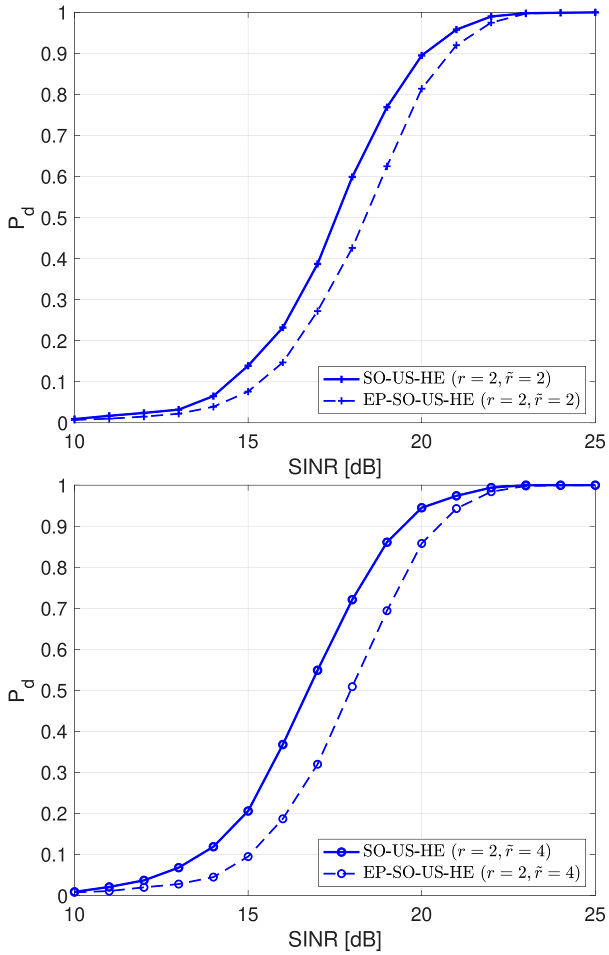

Fig. 1. P_d versus SINR [dB]: matched case ($r = \tilde{r} = 2$) at the top and mismatched case ($r = 2, \tilde{r} = 4$) at the bottom.

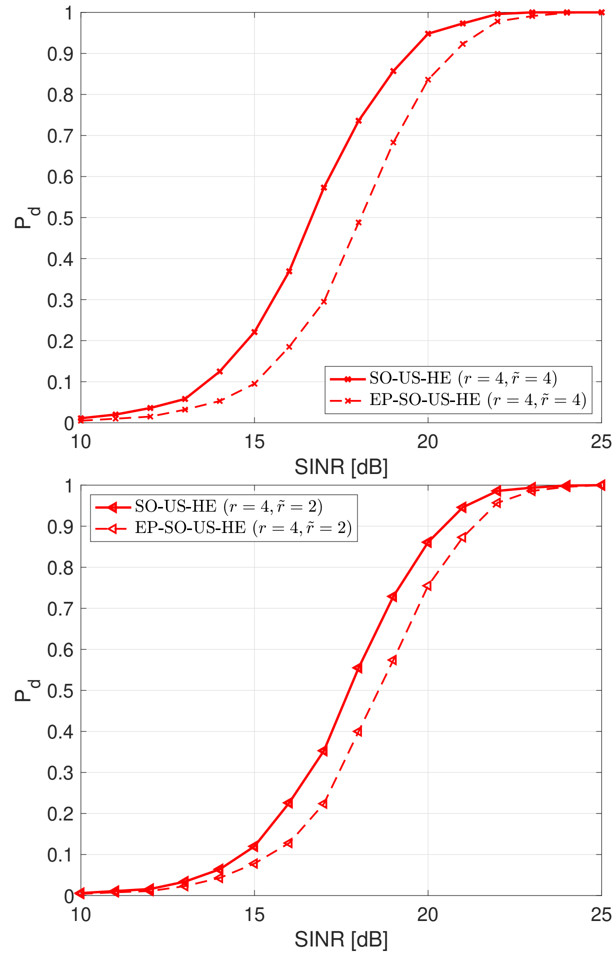

Fig. 2. P_d versus SINR [dB]: matched case ($r = \tilde{r} = 4$) at the top and mismatched case ($r = 4, \tilde{r} = 2$) at the bottom.

V. Conclusions

In this paper, we have analyzed the adaptive detection of Gaussian subspace signals in a homogeneous environment focusing on a second-order signal model. Our study has extended previous analyses by considering cases where the nominal and actual subspace dimensions do not match, as well as scenarios involving NLJ. The impact of the subspace mismatch depends on the scenario: reducing the assumed subspace dimension negatively affects detection performance because part of the signal energy is not exploited in the construction of the decision statistic. In the opposite case, the mismatch might lead to a slight performance improvement due to the collection of more energy. Finally, the detector demonstrated resilience against noise-like jamming, maintaining relatively high detection probabilities even in the presence of strong interference.

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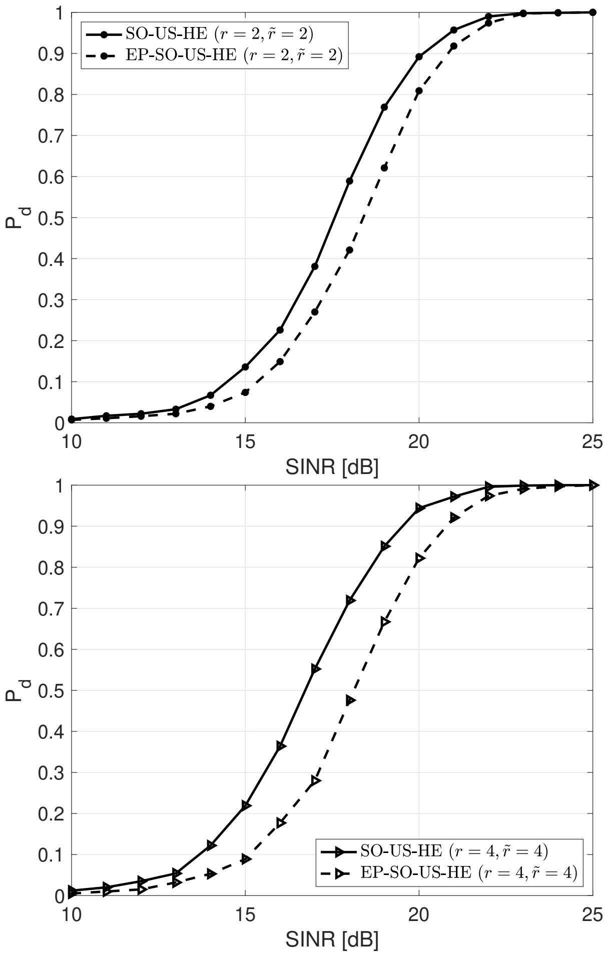

Fig. 3. P_d versus SINR [dB] with NLJ: $r = \tilde{r} = 2$ at the top and $r = \tilde{r} = 4$ at the bottom.

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