ESTIMATING A MIXTURE OF SINUSOIDS WITH MULTI-CHANNEL UNLIMITED SAMPLING

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ABSTRACT

This paper considers the estimation of a mixture of sinusoids in an unlimited sensing framework. A modulo analog-to-digital converter (ADC) is employed to fold back the input signal into a bounded interval before samples are token. We show that, for a band-limited signal, when the sampling rate satisfies a certain condition that is closely related to the dynamic range of the modulo ADC, the first-order difference of the original samples can be uniquely decomposed as a sum of the first-order difference of the modulo samples and a constant with only three possible values. This enables us to formulate the problem of estimating a mixture of sinusoids as a joint sparse signal recovery and unknown integer parameters estimation problem, which can be efficiently solved via a mixed-integer linear program. In addition, a multi-channel sampling architecture is employed to form a "virtual" modulo ADC with an enlarged dynamic range. This improvement helps reduce the sampling rate for estimating sinusoidal mixtures. Numerical simulations are conducted to illustrate the performance of the proposed method.

Index Terms— Parameter estimation, sinusoidal mixture, unlimited sampling, first-order difference, mixed-integer linear program

1. INTRODUCTION

Frequency and amplitude estimation for a mixture of sinusoids from a finite number of samples can be found in various applications, such as ranging and direction finding [1]. Existing solutions to this problem include traditional methods like the MUSIC algorithm [2] and compressive sensing based approaches [3]. For practical implementation, an analog-to-digital converter (ADC) is employed to obtain samples of the sinusoidal signal. However, clipping/saturation occurs when the amplitude of the input signal exceeds the dynamic range of the conventional ADC, resulting in potentially severe information loss and significant performance degradation of such solutions.

The recently introduced unlimited sensing framework is a promising approach to addressing the above drawback of conventional ADCs with a limited dynamic range [4]. Specifically, a modulo ADC (M-ADC) is employed to map the input signal to a bounded interval $[-\lambda, \lambda]$ where λ is the maximum dynamic range of the M-ADC. This operation is mathematically equivalent to taking modulo of the input signal with respect to λ, which eliminates the saturation problem of conventional sampling systems. However, the modulo operation is essentially a nonlinear mapping from the input to the output, which creates a challenge in signal processing. The problem has been extensively investigated over the past few years, leading to a multitude of methods including the mixture model based solutions [5–7] and the higher-order difference (HOD) based solutions [8–11]. These solutions, however, are sensitive to noise.

To tackle this issue, we examine the properties of the first-order difference of modulo samples and formulate the parameter estimation of a mixture of sinusoids as a joint sparse vector recovery and unknown constant parameter estimation problem. By introducing two auxiliary vectors, the joint estimation is recast as a mixed-integer linear program (MILP) which can be efficiently solved via a branch-and-bound algorithm [12]. In addition, we further introduce a multi-channel modulo sampling architecture to further relax the requirements on sampling intervals. Simulation results show that the proposed method outperforms the high-order difference based approach, while the multi-channel sampling scheme significantly reduces the required conditions for sampling intervals.

2. PROBLEM FORMULATION

Consider the following sinusoidal mixture model

$$y(t) = \sum_{k=1}^{K} \alpha_k e^{-j\omega_k t} \tag{1}$$

where $t \in \mathbb{R}$ denotes the time, K is the number of sinusoidal components, and $\{\alpha_k, \omega_k\}$ are respectively, the complex amplitude and frequency associated with the kth component.

This research was supported by the national natural science foundation of China under Grants No. 62103083.

The sinusoidal mixture signal is sampled via a M-ADC with sampling interval ΔT, resulting in the following discrete measurement

$$z_m \triangleq \mathcal{U}_\lambda(y_m) + v_m \tag{2}$$

where $y_m \triangleq y(m\Delta T)$, m is the sampling index, $v_m \sim \mathbb{CN}(0, \sigma^2)$ is the complex additive white Gaussian noise, and $\mathcal{U}_\lambda(\cdot)$ is the modulo operation performed on a complex value, which is defined as

$$\mathcal{U}_\lambda(a) \triangleq \mathcal{M}_\lambda(a^r) + j\mathcal{M}_\lambda(a^i) \tag{3}$$

where a^r and a^i respectively denote the real and imaginary parts of the complex argument a, and $\mathcal{M}_\lambda$ is a nonlinear mapping in the real domain

$$\mathcal{M}_\lambda : b \mapsto 2\lambda \left(\left\langle \frac{b}{2\lambda} + \frac{1}{2} \right\rangle - \frac{1}{2} \right) \tag{4}$$

in which $\langle b \rangle \triangleq b - \lfloor b \rfloor$ is the fractional part of b and $\lambda \geq 0$ is the operation range of the M-ADC. The M-ADC maps an arbitrary value into its dynamic range $[-\lambda, \lambda]$, which avoids the saturation/clipping in conventional ADCs.

The objective is to estimate $\{\alpha_k, \omega_k\}$ from the modulo measurements $\{z_m\}_{m=1}^M$. To this end, we employ the compressive sensing framework and discretize the continuous frequency parameter space into a finite set of grid points, say P ($P \gg K$) points in total. Define $\boldsymbol{a}_p \triangleq [e^{-j\omega_p \Delta T} \ \cdots \ e^{-j\omega_p M\Delta T}]^T$, $\boldsymbol{A} \triangleq [\boldsymbol{a}_1 \ \cdots \ \boldsymbol{a}_P]$, $\boldsymbol{\alpha} \triangleq [\alpha_1 \ \cdots \ \alpha_P]^T$, $\boldsymbol{y} \triangleq [y_1 \ \cdots \ y_M]^T$, $\boldsymbol{z} \triangleq [z_1 \ \cdots \ z_M]^T$, and $\boldsymbol{v} \triangleq [v_1 \ \cdots \ v_M]^T$. We can then rewrite (2) as

$$\boldsymbol{z} = \mathcal{U}_\lambda(\boldsymbol{y}) + \boldsymbol{v} = \mathcal{U}_\lambda(\boldsymbol{A}\boldsymbol{\alpha}) + \boldsymbol{v} \tag{5}$$

where $\boldsymbol{\alpha}$ is a K-sparse vector. Here we assume that the unknown frequency components lie on the discretized grid. It is apparent from (5) that we cannot directly apply the compressive sensing framework to estimate $\{\alpha_k, \omega_k\}$. The major reason is that, due to the nonlinear mapping involved in the modulo sampling, (5) is no longer a linear expression. To deal with this issue, one can employ the uniform decomposition of a modulo sample [4], i.e., $\mathcal{U}_\lambda(\boldsymbol{A}\boldsymbol{\alpha})$ can be reinteperated as

$$\mathcal{U}_\lambda(\boldsymbol{A}\boldsymbol{\alpha}) = \boldsymbol{A}\boldsymbol{\alpha} - 2\lambda\boldsymbol{e} \tag{6}$$

where $\boldsymbol{e} \in \mathbb{C}^{M\times 1}$ is a complex vector whose real and imaginary parts are integers. Leveraging (6) we can formulate the problem of estimating a mixture of sinusoids as the following optimization problem:

$$\begin{aligned} \min_{\boldsymbol{\alpha}, \boldsymbol{e}} \quad & \|\boldsymbol{\alpha}\|_0 \\ \text{s.t.} \quad & \|\boldsymbol{z} - \boldsymbol{A}\boldsymbol{\alpha} - 2\lambda\boldsymbol{e}\|_2^2 < \epsilon, \ \boldsymbol{e}^r \in \mathbb{Z}^M, \ \boldsymbol{e}^i \in \mathbb{Z}^M \end{aligned} \tag{7}$$

where $\|\cdot\|_0$ is the ℓ_0-norm which counts the number of nonzero components, ϵ is a user-defined error tolerance, $\{\boldsymbol{e}^r, \boldsymbol{e}^i\}$ are the real and imaginary parts of the complex vector $\boldsymbol{e}$, and $\mathbb{Z}^M$ represents the M-dimensional integer space. Such an optimization problem can be approximated by a MILP and solved via the branch-and-bound algorithm [12]. This approach, however, is not suitable when λ is significantly smaller than the real dynamic range of y_m, which leads to a large *folding number* [4]. This is because the search space increases exponentially with the folding number. To reduce the search space, in the following, we introduce a first-order difference based solution.

3. PROPOSED SOLUTION VIA MILP

Define the first-order difference of the original sample as $\tilde{y}_m \triangleq y_{m+1} - y_m$ and the first-order difference of the modulo sample as $\tilde{z}_m \triangleq z_{m+1} - z_m$. It can be readily verified that

$$\tilde{z}_m = \tilde{y}_m + 2\lambda\tilde{e}_m \tag{8}$$

where $\tilde{e}_m \triangleq e_{m+1} - e_m$ is also an integer. When the sampling rate is sufficiently large, two successive samples of the original signal undergo slow variations, which implies that e_{m+1} and e_m are very close, and the absolute value of $\tilde{e}_m$ is relatively small. Furthermore, we consider the case that both the real and imaginary parts of $\tilde{e}_m$ belong to the set $\{0, \pm 1\}$. The sampling interval to guarantee this will be discussed later.

We have the following theorem related to the sampling interval to guarantee $\tilde{e}_m$ belong to the set $\{0, \pm 1\}$.

Theorem 1. *Define* $\omega \triangleq \max\{\omega_1, \cdots, \omega_k\}$, $\bar{\beta} \triangleq \sum_{k=1}^K |\beta_k \bar{\omega}_k|$, *and* $\bar{\omega}_k = \omega_k/\omega$. *If the sampling interval satisfies*

$$\Delta T < \frac{1}{\omega} \left(\frac{\bar{\beta}}{2\lambda} \right)^{-1} \tag{9}$$

then we have $z_m - y_m = 2\lambda e_m$, *where* e_m *is a complex number whose real part and imaginary part belong to the set* $\{0, \pm 1\}$.

The proof of Theorem 1 can be readily derived from Theorem 2 of [13], and is omitted due to space limitation.

Define $\tilde{\boldsymbol{z}} \triangleq [\tilde{z}_1, \cdots, \tilde{z}_{M-1}]^T$ and $\tilde{\boldsymbol{y}} \triangleq [\tilde{y}_1, \cdots, \tilde{y}_{M-1}]^T$. The relationship between $\boldsymbol{z}$ and $\tilde{\boldsymbol{z}}$, as well as $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$, can be written as

$$\tilde{\boldsymbol{z}} = \boldsymbol{D}\boldsymbol{z}, \ \tilde{\boldsymbol{y}} = \boldsymbol{D}\boldsymbol{y} \tag{10}$$

where $\boldsymbol{D} \in \mathbb{R}^{(M-1)\times M}$ with its (i,j)th element given by $\delta_{j-i-1} - \delta_{j-i}$ where δ_i is the Kronecker delta function. Furthermore, defining $\tilde{\boldsymbol{e}} \triangleq [\tilde{e}_1, \cdots, \tilde{e}_{M-1}]^T$ and using (8), we can obtain

$$\begin{aligned} \tilde{\boldsymbol{z}} &= \tilde{\boldsymbol{y}} + 2\lambda\tilde{\boldsymbol{e}} = \boldsymbol{D}\boldsymbol{A}\boldsymbol{\alpha} + 2\lambda\tilde{\boldsymbol{e}} \\ &\triangleq \bar{\boldsymbol{A}}\boldsymbol{\alpha} + 2\lambda\tilde{\boldsymbol{e}} \end{aligned} \tag{11}$$

where $\bar{\boldsymbol{A}} \triangleq \boldsymbol{D}\boldsymbol{A}$. Thus we can formulate the estimation problem of a mixture of sinusoids as

$$\begin{aligned} \min_{\boldsymbol{\alpha},\tilde{\boldsymbol{e}}} \quad & \|\boldsymbol{\alpha}\|_0 \\ \text{s.t.} \quad & \|\tilde{\boldsymbol{z}} - \bar{\boldsymbol{A}}\boldsymbol{\alpha} - 2\lambda\tilde{\boldsymbol{e}}\|_2^2 \leq \epsilon \\ & \tilde{\boldsymbol{e}}^r, \tilde{\boldsymbol{e}}^i \in \{0, \pm 1\}^{(T-1)} \end{aligned} \tag{12}$$

For convenience, we write (12) into a real-valued form, i.e.,

$$\begin{aligned} \min_{\breve{\boldsymbol{\alpha}},\breve{\boldsymbol{e}}} \quad & \|\breve{\boldsymbol{\alpha}}\|_0 \\ \text{s.t.} \quad & \|\breve{\boldsymbol{z}} - \breve{\boldsymbol{A}}\breve{\boldsymbol{\alpha}} - 2\lambda\breve{\boldsymbol{e}}\|_2^2 \leq \epsilon \\ & \breve{\boldsymbol{e}} \in \{0, \pm 1\}^{2(T-1)} \end{aligned} \tag{13}$$

where we define $\breve{\boldsymbol{\alpha}} \triangleq [(\boldsymbol{\alpha}^r)^T \ (\boldsymbol{\alpha}^i)^T]^T$, $\breve{\boldsymbol{e}} \triangleq [(\tilde{\boldsymbol{e}}^r)^T \ (\tilde{\boldsymbol{e}}^i)^T]^T$, $\breve{\boldsymbol{z}} \triangleq [(\tilde{\boldsymbol{z}}^r)^T \ (\tilde{\boldsymbol{z}}^i)^T]^T$, and

$$\breve{\boldsymbol{A}} \triangleq \begin{bmatrix} \bar{\boldsymbol{A}}^r & -\bar{\boldsymbol{A}}^i \\ \bar{\boldsymbol{A}}^i & \bar{\boldsymbol{A}}^r \end{bmatrix}$$

Unfortunately, (13) is a mixed-integer quadratic optimization problem with ℓ_0-norm, which is in general intractable. To deal with this challenge, we consider the ℓ_1-norm convex relaxation of the ℓ_0-norm with respect to $\boldsymbol{\alpha}$ and replace the quadratic inequality constraint by a linear constraint, resulting in the following optimization problem

$$\begin{aligned} \min_{\breve{\boldsymbol{\alpha}},\breve{\boldsymbol{e}}} \quad & \|\breve{\boldsymbol{\alpha}}\|_1 \\ \text{s.t.} \quad & \epsilon'\mathbf{1} \preceq \left(\breve{\boldsymbol{z}} - \breve{\boldsymbol{A}}\breve{\boldsymbol{\alpha}} - 2\lambda\breve{\boldsymbol{e}}\right) \preceq \epsilon'\mathbf{1} \\ & \breve{\boldsymbol{e}} \in \{0, \pm 1\}^{2(T-1)} \end{aligned} \tag{14}$$

where ϵ' is a user-defined parameter and $\preceq$ denotes the element-wise inequality. For the pth component of $\breve{\boldsymbol{\alpha}}$, i.e., $\breve{\alpha}_p$, we define two auxiliary variables $\xi_p = \max\{\breve{\alpha}_p, 0\}$ and $\zeta_p = \max\{-\breve{\alpha}_p, 0\}$ such that

$$\breve{\alpha}_p = \xi_p - \zeta_p, \quad |\breve{\alpha}_p| = \xi_p + \zeta_p \tag{15}$$

Therefore we can re-express $\breve{\boldsymbol{\alpha}}$ and $\|\breve{\boldsymbol{\alpha}}\|_1$ as

$$\breve{\boldsymbol{\alpha}} = \boldsymbol{\xi} - \boldsymbol{\zeta}\ , \quad \|\breve{\boldsymbol{\alpha}}\|_1 = \mathbf{1}^T(\boldsymbol{\xi} + \boldsymbol{\zeta}) \tag{16}$$

where $\boldsymbol{\xi} \in \mathbb{R}_+^P$ and $\boldsymbol{\zeta} \in \mathbb{R}_+^P$ are vectors with ξ_p and ζ_p being their pth component respectively, and $\mathbb{R}_+$ represents the non-negative real numbers. Such a representation leads to

$$\begin{aligned} \min_{\boldsymbol{\xi},\boldsymbol{\zeta},\breve{\boldsymbol{e}}} \quad & \mathbf{1}^T(\boldsymbol{\xi} + \boldsymbol{\zeta}) \\ \text{s.t.} \quad & \epsilon'\mathbf{1} \preceq \left(\breve{\boldsymbol{z}} - \breve{\boldsymbol{A}}\left(\boldsymbol{\xi} - \boldsymbol{\zeta}\right) - 2\lambda\breve{\boldsymbol{e}}\right) \preceq \epsilon'\mathbf{1} \\ & \breve{\boldsymbol{e}} \in \{0, \pm 1\}^{2(T-1)},\ \boldsymbol{\xi} \in \mathbb{R}_+^P,\ \boldsymbol{\zeta} \in \mathbb{R}_+^P \end{aligned} \tag{17}$$

The problem (17) is a MILP that can be solved via the branch-and-bound algorithm (e.g., the off-the-shelf tool *intlinprog* in Matlab). Once we obtain $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, we can construct $\breve{\boldsymbol{\alpha}} = \boldsymbol{\xi} - \boldsymbol{\zeta}$.

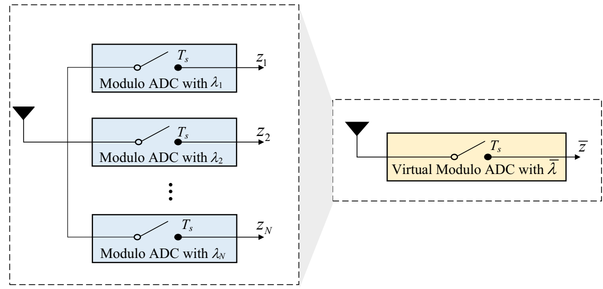

Fig. 1: The "viritual" modulo ADC framework

4. MULTI-CHANNEL UNLIMITED SAMPLING

According to Theorem 1, the upper bound of the required sampling interval is controlled by three parameters, i.e., ω, $\bar{\beta}$ and λ. The first two parameters are application-dependent variables, making it impractical to adjust them to relax the strict requirement for ΔT. The only parameter we can change is λ, which denotes the dynamic range of the M-ADC. However, increasing the dynamic range of the M-ADC introduces several serious issues, such as reduced resolution, increased power consumption, and increased hardware complexity.

Recently, inspired by the Chinese remainder theorem (CRT) and multi-channel sampling techniques, a multi-channel modulo sampler was introduced in [14, 15]. Specifically, as illustrated in Fig. 3, the multi-channel sampler consists of several sampling branches, with each branch equipped with a M-ADC. It turns out that the multi-channel M-ADC sampling architecture is equivalent to a single "virtual" M-ADC (VM-ADC). Let $\bar{\lambda}$ denote the measurement range of the VM-ADC. Apparently, $\bar{\lambda}$ can be obtained via

$$\bar{\lambda} = \text{lcm}\left(\lambda_1, \lambda_2, \cdots, \lambda_N\right) \tag{18}$$

where lcm() denotes the least common multiple. Therefore, to ensure that the VM-ADC has the largest possible dynamic range, the integers $\{\lambda_i\}_{i=1}^N$ should be co-prime.

In addition, the output of the VM-ADC, i.e., $\bar{z}$, should be calculated based on $\{\lambda_i, z_i\}_{i=1}^N$. Theoretically, $\bar{z}$ can be determined using CRT when the outputs of each sampling branch are noiseless. However, in real applications, the measurement noise is inevitable. Directly computing $\bar{z}$ via CRT may lead to significant errors, as CRT is sensitive to noise. Several robust CRTs were proposed to deal with noise [14, 16, 17]. However, these methods either require a high signal to noise ratio (SNR) or involve a high computational complexity. In this work, we propose a simple but efficient searching approach to calculate the output of the VM-ADC.

Since the dynamic range of the VM-ADC is $\bar{\lambda}$, $\bar{z}$ should be within $[-\bar{\lambda}, \bar{\lambda}]$. Therefore, $\bar{z}$ can be solved by the following optimization problem

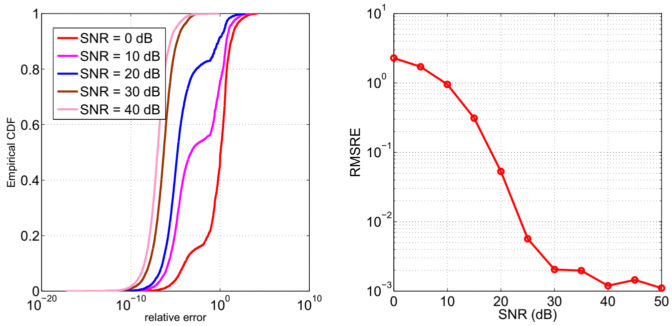

Fig. 2: The empirical CDF of relative error (left) as well as the RMSRE of the VM-ADC (right).

$$\bar{z} = \arg \min_{\bar{z} \in [-\bar{\lambda}, \bar{\lambda}]} \sum_{i=1}^{N} \|z_i - \mathcal{U}_{\lambda_i}(\bar{z})\|_2 \tag{19}$$

Directly solving the above optimization problem is intractable due to the nonlinear modulo operation. To address this challenge, we can employ an alternative search strategy. Specifically, we define the search step as δ and discretize the interval $[-\bar{\lambda}, \bar{\lambda}]$ into a finite set of gird points. We then evaluate the cost functions for each element in this finite set and set $\bar{z}$ as the element with the smallest cost function. The details are summarized in Algorithm 1.

Algorithm 1 Obtaining the output of the VM-ADC

Input: $\{\lambda_i, z_i\}_{i=1}^N$, $\bar{\lambda}$, and δ.
Initialized: $\bar{z} = z_s = 0$, $\mathcal{L} = \sum_{i=1}^N \|z_i - \mathcal{U}_{\lambda_i}(z_s)\|_2$.
repeat
    Compute $z_s = z_s + \delta$ and $\mathcal{L}' = \sum_{i=1}^N \|z_i - \mathcal{U}_{\lambda_i}(z_s)\|_2$;
    if $\mathcal{L}' < \mathcal{L}$ **then**
        $\bar{z} = z_s$, $\mathcal{L} = \mathcal{L}'$;
    end if
until $z_s > \bar{\lambda}$;
Output: $\bar{z} = z_s$.

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5. SIMULATION RESULTS

In this section, simulation results are presented to illustrate the performance of the proposed solutions. We first evaluate the performance of Algorithm 1. We consider three sampling branches, with the dynamic ranges of these associated M-ADCs set to 3, 5, and 7, respectively. This leads to a VM-ADC with $\bar{\lambda} = 105$. The noise observed at the ith sampling branch is disturbed as $\mathbb{N}(0, \sigma_i^2)$. The SNR of each sampling branch, defined as $10 \log\left((z_i^2)/\sigma_i^2\right)$, is set equal. The test input signal (denoted by y) is uniformly distributed with the interval $[-1000, 1000]$, and hence the noise-free modulo sample (denoted by z) is given by $z = \mathcal{U}_{105}(y)$. The left figure of Fig. 2 shows the empirical cumulative distribution function (CDF) of the relative error for 10000 realizations of y under different SNRs, while the right figure of Fig. 2 plots the root mean square relative error (RMSRE), defined as RMSRE $\triangleq \sqrt{\mathbb{E}((\bar{z} - z)^2/z^2)}$, as a function of SNR. From these figures, we can see that Algorithm 1 recovers the VM-ADC output with a relatively small construction error when SNR is greater than 20 dB.

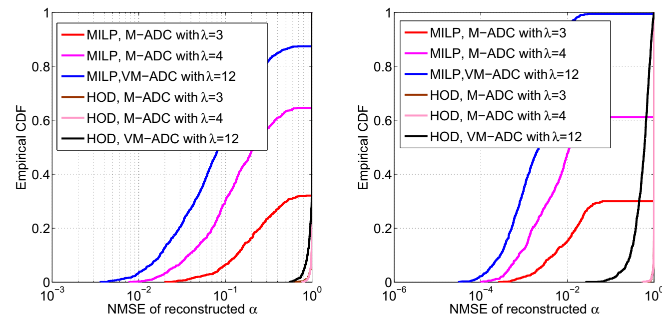

Fig. 3: The empirical CDF of the NMSE of the constructed $\boldsymbol{\alpha}$ when $\Delta T = 0.16$s. Left: SNR = 20dB; Right: SNR = 40dB.

We now present results to illustrate the performance of the proposed mixed-integer linear programming (MILP) based solution compared to the higher-order difference (HOD) based solution. We consider $K = 10$ sinusoidal components, where the magnitude of the complex amplitudes are randomly selected from the interval $(0, 10]$ and the frequencies are randomly selected from the set $\{\omega | \omega = 0.1k\pi, k = 1, \cdots, 20, k \in \mathbb{Z}\}$. The initial phase of each sinusoidal component is randomly selected from $(0, 2\pi]$. The sampling interval ΔT is set to 0.16s, which may fail to meet the required conditions for the MILP-based and HOD-based methods. The SNR of each sampling channel is defined as SNR $= \mathbb{E}(\|\boldsymbol{z}_i\|^2)/\mathbb{E}(\|\boldsymbol{v}_i\|^2)$. We consider two sampling branches with $\lambda_1 = 2$ and $\lambda_2 = 3$, resulting in a VM-ADC with $\bar{\lambda} = 12$.

The empirical CDF of the NMSEs of respective methods are plotted in Fig. 3, where two SNRs are employed. We observe that the MILP-based method outperforms the HOD-based method given the same dynamic range of the M-ADC/VM-ADC. This is because the HOD-based method generally relies on higher-order differences of the modulo samples, which tends to amplify the measurement noise and results in a lower effective SNR. In contrast, the MILP-based solution benefits significantly from the multi-channel sampling scheme, which can substantially relax the required conditions for ΔT, increasing it up to four times larger compared to the ΔT required for each individual sampling branch. This result can be derived from Theorem 1.

6. CONCLUSIONS

In this work, we studied the problem of estimating a mixture of sinusoids in an unlimited sensing framework, which employs M-ADCs to avoid clipping or saturation that may occur in conventional ADCs. We examined the properties of the first-order difference of modulo samples and proposed a novel MILP-based solution that has a reduced search space. In addition, we utilized a multi-channel modulo sampling architecture to form a VM-ADC, which significantly reduces the required sampling frequency. Simulation results show that the proposed MILP-based method not only achieves a significant performance improvement over the HOD based methods but also is robust against noise.

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