

# Cross-Laplacians Based Topological Signal Processing over Cell MultiComplexes

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**Abstract**—The study of the interactions among different types of interconnected systems in complex networks has attracted significant interest across many research fields. However, effective signal processing over layered networks requires topological descriptors of the intra- and cross-layers relationships that are able to disentangle the homologies of different domains, at different scales, according to the specific learning task. In this paper, we present Cell MultiComplex (CMC) spaces, which are novel topological domains for representing multiple higher-order relationships among interconnected complexes. We introduce cross-Laplacians matrices, which are algebraic descriptors of CMCs enabling the extraction of topological invariants at different scales, whether global or local, inter-layer or intra-layer. Using the eigenvectors of these cross-Laplacians as signal bases, we develop topological signal processing tools for CMC spaces. In this first study, we focus on the representation and filtering of noisy flows observed over cross-edges between different layers of CMCs to identify cross-layer hubs, i.e., key nodes on one layer controlling the others.

**Index Terms**—Topological signal processing, cell multicomplexes, cross-Laplacians, multilayer networks, algebraic topology.

## I. INTRODUCTION

In recent years, there has been a growing interest in the study of complex networks, as they model systems where a set of entities interact in different ways through relationships that often convey different meanings and scales [1]. Typically, complex systems are composed of multiple interconnected subsystems organized into distinct layers of connectivity.

Multilayer networks [1], [2], [3] have been extensively studied over the last few decades as they provide a natural and powerful framework for modeling heterogeneous systems. Unlike traditional single-layer networks, multilayer networks model multiple types of interactions within a system, by efficiently describing complex phenomena. For instance, in neuroscience, multimodal brain connectomes can be modeled as multilayer networks [4], where different layers correspond to different modes of brain connectivity, potentially giving more nuances than single layer brain networks [5]. In biological molecular networks [6], multiple biochemical interactions, as protein-gene-metabolite interactions, can be represented by multilayer networks. Similarly, in telecommunication and

transportation networks [7] multilayers networks are efficient tools for analyzing different levels of connectivity. Most of these studies focused on modeling inter- and intra-layers relationships using graphs, which can only capture pairwise interactions between entities. However, in many complex systems, interactions typically involve groups of similar or heterogeneous entities, leading to recent studies on higher-order multiplex networks [8] based on simplicial complexes. Simplicial complexes are topological spaces able to capture higher-order interactions between the elements of a set, while preserving the inclusion property. Despite recent success in topological representations of complex systems via simplicial complexes, the current representation of these spaces rely on algebraic topological descriptors that fail to disentangle the local intra- and inter-layers topological features. In this regard, recently, the authors in [9] introduced an interesting representation of simplicial multi-complex networks using the cross-Laplacians as algebraic topological descriptors. These matrices represent powerful algebraic tools for analyzing both global and local topological invariants of a space, i.e. properties that keep unchanged under homeomorphisms. These topological invariants are encoded by the so called cross-Betti vectors, i.e. a set of cross-Betti numbers able to capture different local topological invariants.

Our first novel contribution in this paper is extending the simplicial complex algebraic representation in [9] to cell complex spaces, which we name *Cell MultiComplexes (CMCs)*. CMCs are powerful spaces capable of capturing multiple interactions of any sparsity order among entities and that can be efficiently represented through cross-Laplacians. By introducing different boundaries maps, cross-Laplacians enable the extraction of different kinds of topological invariants according to the scales we aim to explore: a global perspective, treating the entire complex as a flattened monolayer structure, or a local view, which disentangles the homologies by investigating as the topology of one layer is related to the others.

Our second key contribution is the development of a signal processing framework on CMCs. We first introduce local Hodge decompositions of signals observed on the cells of a CMC, enabling signal spectral representation via low-dimensional dictionaries. In this initial study, our learning-task focuses on processing flows over the cross-edges connecting different layers in order to identify harmonic cross-hubs

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between layers. Then, we show how the homologies of the  $(0,0)$ -cross Laplacians can effectively capture the number of harmonic cross-hubs between layers, i.e. key nodes controlling inter-layers connectivity. Using the eigenvectors of this cross-Laplacian as signal bases, we show how noisy flows across two distinct layers of a CMC can be efficiently filtered to recover signal components relevant for identifying cross-hubs.

## II. CELL MULTICOMPLEXES

In this section we introduce the fundamental notions defining cell multicomplexes. Building on the topological tools developed for simplicial complexes in [9], our first novel contribution is to extend these representation methods to encompass more general topological structures, such as cell complexes. We begin by recalling the notion of cell complexes and then we introduce cell multicomplexes topological spaces.

**Cell complexes.** An abstract cell complex (ACC) [10]  $\mathcal{C} = \{\mathcal{S}, \prec_b, \dim\}$  is a set  $\mathcal{S}$  of abstract elements  $c$ , named cells, provided with a binary relation  $\prec_b$ , called the bounding (or incidence) relation, and with a dimension function, denoted by  $\dim(c)$ , assigning to each  $c \in \mathcal{S}$  a non-negative integer  $[c]$ . A cell  $c$  is called a  $k$ -cell if  $\dim(c) = k$  where  $k$  is the dimension (or order) of  $c$ . We denote a cell of order  $k$  as  $c_k$ . Therefore, 0-cells  $c_0$  are named vertices and 1-cells  $c_1$  edges. We say that the  $k$ -cell  $c_k$  lower bounds the  $(k+1)$ -cell  $c_{k+1}$  if  $c_k \prec_b c_{k+1}$  and  $c_k$  is a face of  $c_{k+1}$ . An ACC is of dimension  $K$  or a  $K$ -dimensional ACC, if the dimensions of all its cells are less than or equal to  $K$ . Given a  $k$ -dimensional cell  $c_k$ , we define its boundary as the set of all cells of dimension less than  $k$  bounding  $c_k$ .

**Cell Multicomplexes.** Let us now introduce the concept of cell multicomplex space.

*Definition 1: A Cell MultiComplex (CMC)  $\mathcal{X}$  is a topological space composed by a finite collection of interdependent abstract cell complexes, each associated with a topological layer. The interdependence among these complexes involve higher-order inter-layer interactions modeled by cross-complexes.*

The inter-layer higher-order interactions are captured by cells of different orders named *cross-cells*. Cross-cells of order 1, 2 and 3 are cross-edges, cross-polygons and cross-polyhedra, respectively. The dimension of a CMC is the maximum order of its cells.

An illustrative example of a CMC composed of  $L = 3$  layers is shown in Fig. 1. It is composed of three intra-layer cell complexes  $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$  interconnected by cross-edges (dashed lines). We can observe three cross-cells of order 2 between layers 1 and 2, one triangle and two squares, and one cross-cell of order 3, a tetrahedron, between layer 2 and 3. Note that CMCs are suitable spaces to represent data observed over higher-order interconnected networks or over different domains each associated with a different layer.

Consider a network with layers indexed according to an increasing order. For simplicity of notation and w.l.o.g., let us assume that cross-cells involve only two layers, as illustrated in Fig. 1. Hence, we denote by  $c_k^{\ell,m}(i)$  the  $i$ -th cross-cell of order  $k \geq 0$ , interconnecting layers  $\ell$  and  $m$  and by  $c_k^\ell$  the intralayer

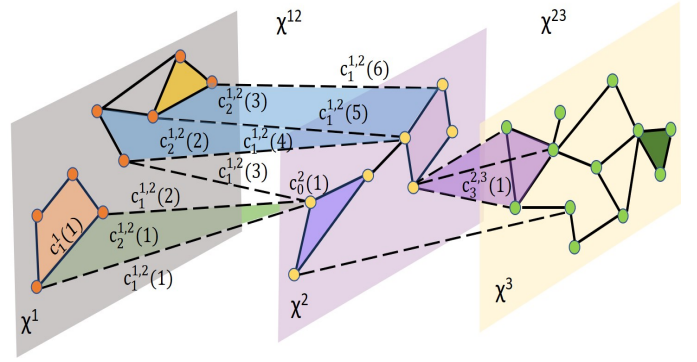


Fig. 1. Illustrative example of a CMC of order 3.

cells, i.e. cells of order  $k$  within layer  $\ell$ . Given the cross-cell  $c_k^{\ell,m}(i)$ , we define its  $\ell$ -layer faces and  $m$ -layer faces as the cells of order  $0 \leq j < k$  that lower bound  $c_k^{\ell,m}(i)$  and belong to the  $\ell$ -layer and  $m$ -layer, respectively. In the example in Fig. 1, the 2-order cross-cell  $c_2^{1,2}(1)$  is a cross-triangle, connecting layers 1 and 2, with lower bounding 1-order cells  $c_1^1(1)$ ,  $c_1^{1,2}(1)$  and  $c_1^{1,2}(2)$ . The face on layer 1 of  $c_2^{1,2}(1)$  is the edge  $c_1^1(1)$ , while the face on layer 2 is the node  $c_0^2(1)$ . Note that cross-cells may have in general faces of different orders on each layer.

A CMC  $\mathcal{X}$  is a collection of intra- and cross-layer complexes. We denote by  $\mathcal{X}^{\ell,m}$  the cross-complex composed of the cross-cells inter-connecting layers  $\ell$  and  $m$ . We further define the cross-complex  $\mathcal{X}_{k,n}^{\ell,m} \subseteq \mathcal{X}^{\ell,m}$  as the collection of cross-cells with faces of order  $k$  in layer  $\ell$  and with faces of order  $n$  in layer  $m$ . Thus, we denote by  $N_{k,n}^{\ell,m}$  the number of cross-cells in  $\mathcal{X}_{k,n}^{\ell,m}$ , i.e.  $N_{k,n}^{\ell,m} = |\mathcal{X}_{k,n}^{\ell,m}|$ . Additionally, we define  $\mathcal{X}_{k,n}^{\ell,m}$  as the intra-layer cell complex of order  $k$  within layer  $\ell$ , using the subscript  $-1$  to indicate the absence of cells over layer  $m$ .

Considering the example in Fig. 1, the cross-complex  $\mathcal{X}^{1,2}$  between layer 1 and 2, is given by  $\mathcal{X}^{1,2} = \{\mathcal{X}_{0,0}^{1,2}, \mathcal{X}_{1,0}^{1,2}, \mathcal{X}_{1,1}^{1,2}\}$  with  $\mathcal{X}_{0,0}^{1,2} = \{c_1^{1,2}(i)\}_{i=1}^6$ ,  $\mathcal{X}_{1,0}^{1,2} = \{c_2^{1,2}(1), c_2^{1,2}(2)\}$ , and  $\mathcal{X}_{1,1}^{1,2} = \{c_2^{1,2}(3)\}$ . Note that the complex  $\mathcal{X}_{0,0}^{1,2}$  is a cross-graph, while  $\mathcal{X}_{1,0}^{1,2}$  and  $\mathcal{X}_{1,1}^{1,2}$  are cross-complexes of order 2.

The orientation of the cross-cells is an ordering choice over its lower bounding cells (see [11], [12]). We use the notation  $c_{k-1}^{\ell,m}(i) \sim c_k^{\ell,m}(j)$  to indicate that the orientation of  $c_{k-1}^{\ell,m}(i)$  is coherent with that of  $c_k^{\ell,m}(j)$  and  $c_{k-1}^{\ell,m}(i) \approx c_k^{\ell,m}(j)$  to indicate opposite orientations. Two  $k$ -order cells are lower adjacent if they share a common face of order  $k-1$  and upper adjacent if they are both faces of a cell of order  $k+1$ .

## III. ALGEBRAIC REPRESENTATION OF CMCs: FROM GLOBAL TO LOCAL INVARIANTS

In many applications, from signal processing to machine learning, data resides on different interconnected networks and the learning task is to uncover global as well as local topological features. Therefore, we need a framework for processing signals defined over cell multicomplexes capable of disentangling the homologies of individual layers and uncovers how one layer influences and controls one another. Depending

on the learning task, we can adopt two main approaches for the analysis of signals over a cell multicomplex. In the first common approach, the entire structure is treated as a monolayer topological domain, so that the Hodge-Laplacian matrix introduced for cell complexes can be used for the representation and processing of signals [12]. In the second, novel approach, we leverage cross-Laplacian matrices for signal representations able to capture intra- and inter-layer homologies and uncover local topological invariants within the complex.

#### A. Cell multicomplex as a monolayer cell-complex

One of the most common approaches in the study of multilayer networks is to represent them as a single monolayer structure. Then, the resulting flattened cell complex can be algebraically represented by using the Hodge-Laplacian matrix [12]. Let us assume that a 2-order multicomplex  $\mathcal{X}$  is composed of  $L$  interconnected layers. The incidence matrix  $\mathbf{B}_k$ , describing which  $k$ -cells are upper adjacent to which  $(k-1)$ -cells is defined as  $B_k(i, j) = 1$  (or  $B_k(i, j) = -1$ ) if  $c_{k-1}(i) \prec_b c_k(j)$  and  $c_{k-1}(i) \sim c_k(j)$  (or  $c_{k-1}(i) \nearrow c_k(j)$ ), while  $B_k(i, j) = 0$  if  $c_{k-1}(i) \not\prec_b c_k(j)$ . Therefore, we can represent the cell multicomplex  $\mathcal{X}$  using the graph Laplacian matrix  $\mathbf{L}_0 = \mathbf{B}_1 \mathbf{B}_1^T$  and the first-order Hodge Laplacian matrix  $\mathbf{L}_1 = \mathbf{B}_1^T \mathbf{B}_1 + \mathbf{B}_2 \mathbf{B}_2^T$  [12].

This representation of the CMC provides global invariants of the topological spaces described by the Betti numbers. Specifically,  $\beta_0 = \dim(\ker(\mathbf{L}_0))$  represents the number of connected components of the multilayer graph, while  $\beta_1 = \dim(\ker(\mathbf{L}_1))$  corresponds to the number of holes in the entire complex, i.e. the number of empty 2-cells within the complex.

#### B. Cross-Laplacians to capture cross-invariants

In this section, we introduce the notion of cross-Laplacians matrices presented in [9] by extending it to cell multicomplexes. For simplicity of notation, let us assume that cross-cells involve only pairs of layers.

**Cross-boundaries maps.** First we introduce the boundaries maps of cross-cells in the perspective of a given layer, i.e. the boundaries maps of cross-cells only with respect to faces belonging to a given layer and keeping fixed all the remaining faces. Let us consider the two layers  $\ell, m$  and denote by  $C_{k,n}$  the real vector space generated by all oriented  $q$ -order  $(k, n)$  cross-cells  $c_q^{\ell, m}$ , with faces of order  $k$  on layer  $\ell$  and faces of order  $n$  on layer  $m$ . To simplify our notation, we omit the dependence of the cell's order  $q$  on the orders  $(k, n)$  of the cells on layers  $\ell$  and  $m$ , respectively. As an example, in a 2-order CMC, for cross-edges, i.e.  $q = 1$ , we have  $(k, n) = (0, 0)$ , while for 2-order cross-cells ( $q = 2$ ) we can have  $(k, n) = (1, 0), (0, 1), (1, 1)$ . Hence, given the cross-complex  $\mathcal{X}_{k,n}^{\ell, m}$  we define two distinct cross-boundaries operators for each cross-cell  $c_q^{\ell, m} \in \mathcal{X}_{k,n}^{\ell, m}$ . The first operator  $\mathbf{B}_{k,n}^{(\ell), m}$  is a boundary map defined with respect to the crossfaces on layer  $\ell$ , while the second operator, denoted as  $\mathbf{B}_{k,n}^{\ell, (m)}$ , is a boundary map with respect to the crossfaces on layer  $m$ . Specifically,  $\mathbf{B}_{k,n}^{(\ell), m} : C_{k,n} \rightarrow C_{k-1,n}$  is the boundary map with respect to

the cells of order  $k$  on layer  $\ell$  as view from layer  $m$ . Thus,  $\mathbf{B}_{k,n}^{(\ell), m}$  is an incidence matrix of dimension  $N_{k-1,n}^{\ell, m} \times N_{k,n}^{\ell, m}$  with entries defined as

$$B_{k,n}^{(\ell), m}(i, j) = \begin{cases} 0, & \text{if } c_{q-1}^{\ell, m}(i) \not\prec_b c_q^{\ell, m}(j) \\ 1, & \text{if } c_{q-1}^{\ell, m}(i) \prec_b c_q^{\ell, m}(j), c_{q-1}^{\ell, m}(i) \sim c_q^{\ell, m}(j) \\ -1, & \text{if } c_{q-1}^{\ell, m}(i) \prec_b c_q^{\ell, m}(j), c_{q-1}^{\ell, m}(i) \nearrow c_q^{\ell, m}(j) \end{cases} \quad (1)$$

where  $c_{q-1}^{\ell, m}(i) \in \mathcal{X}_{k-1,n}^{\ell, m}$  and  $c_q^{\ell, m}(j) \in \mathcal{X}_{k,n}^{\ell, m}$ ,  $\forall i, j$ . Similarly, the matrices  $\mathbf{B}_{k,n}^{\ell, (m)} : C_{k,n} \rightarrow C_{k,n-1}$  of dimension  $N_{k,n-1}^{\ell, m} \times N_{k,n}^{\ell, m}$  are boundaries with respect to crossfaces on layer  $m$  with entries  $B_{k,n}^{\ell, (m)}(i, j)$  defined as in (1), except that  $c_{q-1}^{\ell, m}(i) \in \mathcal{X}_{k,n-1}^{\ell, m}$  and  $c_q^{\ell, m}(j) \in \mathcal{X}_{k,n}^{\ell, m}$ ,  $\forall i, j$ . It can be proved (we omit here the proof for lack of space) that

$$\mathbf{B}_{k,n}^{(\ell), m} \mathbf{B}_{k+1,n}^{(\ell), m} = \mathbf{0} \quad \text{and} \quad \mathbf{B}_{k,n}^{\ell, (m)} \mathbf{B}_{k,n+1}^{\ell, (m)} = \mathbf{0}. \quad (2)$$

As an example, let us consider the cross-complex  $\mathcal{X}_{1,0}^{1,2} = \{c_2^{1,2}(1), c_2^{1,2}(2)\}$  in Fig. 1. This complex consists of two cross-cells of order  $q = 2$ , i.e.,  $c_2^{1,2}(1)$  and  $c_2^{1,2}(2)$ . These two cells have faces of order 1 (edge or paths) on layer 1 and one face of order 0 (vertex) on layer 2. The bounding cells of  $c_2^{1,2}(1)$  are: with respect to cells on layer 1, the two cross-edges  $c_1^{1,2}(1)$  and  $c_1^{1,2}(2)$ , while with respect to cells on layer 2 the bounding cell is  $c_1^1(1)$ .

**Cross-Laplacian matrices.** Given the two layers  $\ell, m$ , we introduce the  $(k, n)$ -cross-Laplacian matrices from layer  $\ell$  as

$$\mathbf{L}_{k,n}^{(\ell), m} = (\mathbf{B}_{k,n}^{(\ell), m})^T \mathbf{B}_{k,n}^{(\ell), m} + \mathbf{B}_{k+1,n}^{(\ell), m} (\mathbf{B}_{k+1,n}^{(\ell), m})^T \quad (3)$$

where the first and second terms encode the lower and upper adjacencies, respectively. Similarly the  $(k, n)$ -cross-Laplacian matrices from layer  $m$  are defined as

$$\mathbf{L}_{k,n}^{\ell, (m)} = (\mathbf{B}_{k,n}^{\ell, (m)})^T \mathbf{B}_{k,n}^{\ell, (m)} + \mathbf{B}_{k,n+1}^{\ell, (m)} (\mathbf{B}_{k,n+1}^{\ell, (m)})^T. \quad (4)$$

These Laplacians matrices are symmetric and semidefinite positive. It can be observed that the intra  $\ell$ -layer Hodge Laplacian of order  $k$  can be derived from (3) by setting  $n = -1$ . Additionally, note that it holds  $\mathbf{B}_{k,-1}^{(\ell), m} = \mathbf{0}$ ,  $\mathbf{B}_{-1,n}^{\ell, (m)} = \mathbf{0}$ ,  $\forall k, n$ . Furthermore, from (2), it can be proved, following similar considerations as in [9], [13], that the space  $\mathbb{R}^{N_{k,n}}$  admits different Hodge decompositions according to the layer from which the boundary is calculated. Specifically, it holds

$$\mathbb{R}^{N_{k,n}} \equiv \text{img}(\mathbf{B}_{k,n}^{(\ell), m})^T \oplus \ker(\mathbf{L}_{k,n}^{(\ell), m}) \oplus \text{img}(\mathbf{B}_{k+1,n}^{(\ell), m}), \quad (5)$$

$$\mathbb{R}^{N_{k,n}} \equiv \text{img}(\mathbf{B}_{k,n}^{\ell, (m)})^T \oplus \ker(\mathbf{L}_{k,n}^{\ell, (m)}) \oplus \text{img}(\mathbf{B}_{k,n+1}^{\ell, (m)}). \quad (6)$$

The orthogonality conditions in (2) enable the definition of the  $(k, n)$ -cross-homology groups of  $\mathcal{X}$  [9], [13], as  $H_{k,n}^{(\ell)} \cong \ker(\mathbf{L}_{k,n}^{(\ell), m})$  and  $H_{k,n}^{\ell, (m)} \cong \ker(\mathbf{L}_{k,n}^{\ell, (m)})$ . The cross-homology groups are determined by their dimensions, named the  $(k, n)$ -cross-Betti numbers [9],  $\beta_{k,n}^{(\ell)} = \dim(\ker(\mathbf{L}_{k,n}^{(\ell), m}))$  and  $\beta_{k,n}^{\ell, (m)} = \dim(\ker(\mathbf{L}_{k,n}^{\ell, (m)}))$ . Then, we can define the  $(k, n)$ -cross-Betti vector of  $\mathcal{X}_{k,n}^{\ell, m}$  as the vector  $\beta_{k,n}^{\ell, m} = [\beta_{k,n}^{(\ell)}, \beta_{k,n}^{\ell, (m)}]$ . These numbers, as we will see for a 2-order CMC, are able to capture the homologies of the intra- and cross-layer cell complexes.

#### IV. SECOND-ORDER CELL MULTICOMPLEXES

Considering a 2-order CMC, we can build different cross-Laplacians according to the topological invariants we aim to detect. In this first study we focus on the  $(0,0)$ -cross-Laplacians. Using (3) the Laplacian  $\mathbf{L}_{0,0}^{(\ell),m}$  is an  $N_{0,0}^{\ell,m} \times N_{0,0}^{\ell,m}$  symmetric matrix indexed on the cross-edges  $c_1^{\ell,m} \in \mathcal{X}_{0,0}^{\ell,m}$  expressed as

$$\mathbf{L}_{0,0}^{(\ell),m} = (\mathbf{B}_{0,0}^{(\ell),m})^T \mathbf{B}_{0,0}^{(\ell),m} + \mathbf{B}_{1,0}^{(\ell),m} (\mathbf{B}_{1,0}^{(\ell),m})^T. \quad (7)$$

Using (1), we get the  $N_{-1,0}^{\ell,m} \times N_{0,0}^{\ell,m}$  incidence matrix

$$B_{0,0}^{(\ell),m}(i, j) = \begin{cases} 0, & \text{if } c_0^m(i) \not\prec_b c_1^{\ell,m}(j) \\ 1, & \text{if } c_0^m(i) \prec_b c_1^{\ell,m}(j), c_0^m(i) \sim c_1^{\ell,m}(j) \\ -1, & \text{if } c_0^m(i) \prec_b c_1^{\ell,m}(j), c_0^m(i) \approx c_1^{\ell,m}(j) \end{cases} \quad (8)$$

with  $c_0^m(i) \in \mathcal{X}_{-1,0}^{\ell,m}$  and  $c_1^{\ell,m}(j) \in \mathcal{X}_{0,0}^{\ell,m}$ . Then, the entry  $(i, j)$  of the lower Laplacian  $(\mathbf{B}_{0,0}^{(\ell),m})^T \mathbf{B}_{0,0}^{(\ell),m}$  is equal to 1 if  $c_1^{\ell,m}(i)$  is lower adjacent to  $c_1^{\ell,m}(j)$  on layer  $m$ , i.e.  $c_1^{\ell,m}(i)$  and  $c_1^{\ell,m}(j)$  have a common vertex on layer  $m$ . The incidence matrix  $\mathbf{B}_{1,0}^{(\ell),m} : C_{1,0} \rightarrow C_{0,0}$ , in the second term of (7), is a  $N_{0,0}^{\ell,m} \times N_{1,0}^{\ell,m}$  matrix with  $N_{1,0}^{\ell,m}$  being the number of cross-cells of order 2 between layer  $\ell, m$  having edges in layer  $\ell$  and one vertex in layer  $m$ . Then, we get from (1):

$$B_{1,0}^{(\ell),m}(i, j) = \begin{cases} 0, & \text{if } c_1^{\ell,m}(i) \not\prec_b c_2^{\ell,m}(j) \\ 1, & \text{if } c_1^{\ell,m}(i) \prec_b c_2^{\ell,m}(j), c_1^{\ell,m}(i) \sim c_2^{\ell,m}(j) \\ -1, & \text{if } c_1^{\ell,m}(i) \prec_b c_2^{\ell,m}(j), c_1^{\ell,m}(i) \approx c_2^{\ell,m}(j) \end{cases} \quad (9)$$

for  $c_1^{\ell,m}(i) \in \mathcal{X}_{0,0}^{\ell,m}$  and  $c_2^{\ell,m}(j) \in \mathcal{X}_{1,0}^{\ell,m}$ . The upper Laplacian  $\mathbf{B}_{1,0}^{(\ell),m} (\mathbf{B}_{1,0}^{(\ell),m})^T$  identifies the upper adjacencies of the cross-edges  $c_1^{\ell,m}$  as boundaries of 2-order cells with edges on layer  $\ell$  and one vertex on layer  $m$ . Similar derivations can be followed to obtain the cross-Laplacian  $\mathbf{L}_{0,0}^{\ell,(m)}$ .

**The Cross-Betti vector**  $\beta_{0,0}$ . To describe the topological invariants encoded by the cross-Betti vector  $\beta_{0,0}$ , we need to introduce the concept of cones [9]. Cones are the shortest paths of length two between nodes within one layer, passing through a node on the other layer and not belonging to the cross-boundary of 2-order cross-cells. For example, a cone can have one vertex on layer  $m$  and the remaining vertices on layer  $\ell$ . The cones are called closed if they form a cycle with intra-layer edges. A cone can also be open, meaning that a vertex on one layer connects clusters on the other layer that are unconnected. Then, the cross-Betti number  $\beta_{0,0}^{(\ell)} = \dim(\ker(\mathbf{L}_{0,0}^{(\ell),m}))$  is equal to the number of cones (closed and open), with one vertex on layer  $m$ , that are not boundaries of 2-order cross-cells. The vertices of the cones on layer  $m$  are named harmonic cross-hubs. Similarly,  $\beta_{0,0}^{(m)} = \dim(\ker(\mathbf{L}_{0,0}^{\ell,(m)}))$  identify the number of cones with one vertex on layer  $\ell$ .

#### V. SIGNAL PROCESSING OVER CMCs

Algebraic representations of CMCs derived from cross-Laplacians offer suitable bases for the processing of signals defined over CMCs. Let us consider a 2-order CMC  $\mathcal{X} = (\mathcal{V}, \mathcal{E}, \mathcal{C})$ , with  $|\mathcal{V}| = N$ ,  $|\mathcal{E}| = E$  and  $|\mathcal{C}| = C$  the

dimension of the node, edges and 2-cells sets, respectively. We can define signals over the set of nodes, edges and 2-cells as  $s_0 : \mathcal{V} \rightarrow \mathbb{R}^N$ ,  $s_1 : \mathcal{E} \rightarrow \mathbb{R}^E$  and  $s_2 : \mathcal{C} \rightarrow \mathbb{R}^C$ , respectively. Using the Hodge decompositions in (5), (6), we can split these signals in different components belonging to orthogonal subspaces and capturing distinct space invariants.

Let us focus on the  $(0,0)$ -cross Laplacian in (7). Then, it follows from (5) that the cross-edges signal  $s_1^{\ell,m}$ , belonging to the space  $\mathbb{R}^{N_{0,0}^{\ell,m}}$ , can be decomposed as

$$s_1^{\ell,m} = \mathbf{B}_{0,0}^{(\ell),mT} s_0^m + \mathbf{B}_{1,0}^{(\ell),m} s_2^{\ell,m} + s_{1,H}^{\ell,m}, \quad (10)$$

where the node signal  $s_0^m \in \mathbb{R}^{N_{-1,1}^{\ell,m}}$  is observed over the nodes within layer  $m$  and  $s_2^{\ell,m} \in \mathbb{R}^{N_{1,0}^{\ell,m}}$  is a 2-order signal observed over  $(1,0)$  cross-cells between layers  $\ell, m$ , i.e. filled cones with one vertex on layer  $m$ . The first term  $\mathbf{B}_{0,0}^{(\ell),mT} s_0^m$  is a flow on the cross-edges with zero-circulation along the cross-edges of 2-order  $(1,0)$  cross-cells. The second flow  $\mathbf{B}_{1,0}^{(\ell),m} s_2^{\ell,m}$  has zero-sum on the vertices over layer  $m$ . Finally, the harmonic edge signal  $s_{1,H}^{\ell,m}$  belongs to the subspace spanned by  $\ker(\mathbf{L}_{0,0}^{\ell,m})$ , whose dimension is the number of cones between the two layers. Note that for the flows between layers  $\ell$  and  $m$ , we can define the term  $\text{div}^{(\ell),m}(s_1^{\ell,m}) = \mathbf{B}_{0,0}^{(\ell),m} s_1^{\ell,m}$ , which is a node signal measuring the conservation of the cross-flows over the nodes of layer  $m$ , while  $\text{curl}^{(\ell),m}(s_1^{\ell,m}) = \mathbf{B}_{1,0}^{(\ell),mT} s_1^{\ell,m}$  is a measure of the flow conservation along cross-edges bounding 2-order  $(1,0)$  cross-cells.

Extending the cell complex spectral theory [12] to CMC and given the eigendecomposition  $\mathbf{L}_{0,0}^{(\ell),m} = \mathbf{U}_{0,0}^{(\ell),m} \mathbf{\Lambda}_{0,0}^{(\ell),m} \mathbf{U}_{0,0}^{(\ell),mT}$ , we can define the CMC Fourier Transform as the projection of a cross-edge signal  $s_1^{\ell,m}$  onto the space spanned by the eigenvectors of  $\mathbf{L}_{0,0}^{(\ell),m}$ , i.e.  $\hat{s}_1^{\ell,m} := \mathbf{U}_{0,0}^{(\ell),mT} s_1^{\ell,m}$ . Hence, the cross-edge signal can be represented as  $s_1^{\ell,m} := \mathbf{U}_{0,0}^{(\ell),m} \hat{s}_1^{\ell,m}$ . In the following, we design optimal signal estimators from observed noisy cross-signals  $y_1^{\ell,m} = s_1^{\ell,m} + n_1$ , where  $n_1$  is additive noise. The optimal node, 2-cells and harmonic signals, can be derived as the solutions of the following problem

$$\begin{aligned} \min_{\substack{s_0^m \in \mathbb{R}^{N_m}, s_2^{\ell,m} \in \mathbb{R}^{N_{1,0}^{\ell,m}}, \\ s_{1,H}^{\ell,m} \in \mathbb{R}^{N_{0,0}^{\ell,m}}}} & \| \mathbf{B}_{0,0}^{(\ell),mT} s_0^m + \mathbf{B}_{1,0}^{(\ell),m} s_2^{\ell,m} + s_{1,H}^{\ell,m} - y_1^{\ell,m} \|^2 \\ \text{s.t.} & \mathbf{B}_{0,0}^{(\ell),m} s_{1,H}^{\ell,m} = 0, \quad \mathbf{B}_{1,0}^{(\ell),mT} s_{1,H}^{\ell,m} = 0. \end{aligned}$$

It can be easily proved that this problem admits the following closed-form solutions [14]:

$$\begin{aligned} \tilde{s}_0^m &= (\mathbf{B}_{0,0}^{(\ell),m} \mathbf{B}_{0,0}^{(\ell),mT})^\dagger \mathbf{B}_{0,0}^{(\ell),m} y_1^{\ell,m} \\ \tilde{s}_2^{\ell,m} &= (\mathbf{B}_{1,0}^{(\ell),mT} \mathbf{B}_{1,0}^{(\ell),m})^\dagger \mathbf{B}_{1,0}^{(\ell),mT} y_1^{\ell,m} \\ \tilde{s}_{1,H}^{\ell,m} &= y_1^{\ell,m} - \mathbf{B}_{0,0}^{(\ell),mT} \tilde{s}_0^m - \mathbf{B}_{1,0}^{(\ell),m} \tilde{s}_2^{\ell,m} \end{aligned} \quad (11)$$

where  $^\dagger$  denotes the Moore-Penrose pseudo-inverse. As numerical example, we consider the two communication networks illustrated in Fig. 2, where nodes represent devices emitting

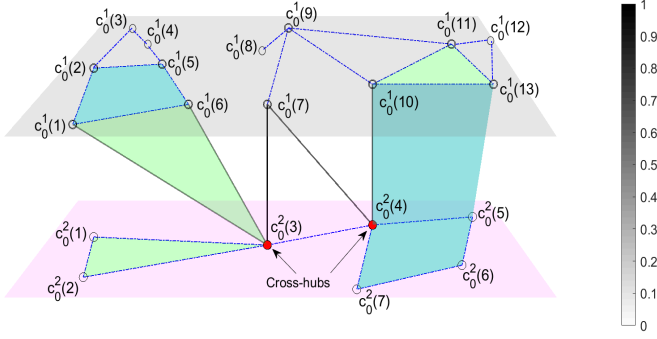


Fig. 2. Recovered harmonic cross-edge signals.

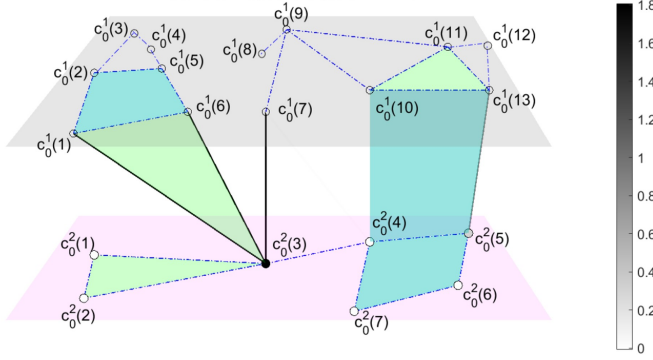


Fig. 3. Recovered cross-edge (zero-circulation) signal and node signal.

data flow packets. The two networks are connected through a set of cross-edges, with some nodes having control functionalities. Our goal is to recover from noisy observations the flows over the cross-edges between the two networks and identify harmonic cross-hubs. Considering the cross-Laplacian matrix  $\mathbf{L}_{0,0}^{(\ell),m}$ , we estimate the cross-edge signals using the closed-forms in (11). In Fig. 2 we represent the intensity of the recovered harmonic signal  $\tilde{s}_{1,H}^{1,2}$  over the cross-edges between the two networks. It can be observed as the harmonic signals tend to be highest on the cross-edges surrounding the two cones identified by the nodes  $(c_0^1(6), c_0^2(3), c_0^1(7))$  and  $(c_0^1(7), c_0^2(4), c_0^1(10))$ . The harmonic cross-hubs are the nodes  $c_0^2(3)$  and  $c_0^2(4)$ , albeit the first cross-hub  $c_0^2(3)$  has a key role in the inter-connectivity between the two networks, as it controls two clusters of nodes on the first network. Furthermore, removing  $c_0^2(3)$  the two clusters on layer 1 are disconnected. To evaluate the activity of the cross-hubs, we derive from the estimated edge signal  $\tilde{s}_1^{\ell,m}$ , the signal  $\text{div}^{(\ell),m}(\tilde{s}_1^{\ell,m})$ . Then, in Fig. 3 we represent over the cross-edges the intensity of the estimated signals  $\mathbf{B}_{0,0}^{(1),2T} \tilde{s}_0^2$ , and, on the nodes of the lower layer, the signal  $\text{div}^{(1),2}(\tilde{s}_1^{1,2})$ . The intensities of the signals are encoded by the colors of cross-edges and nodes (in grayscale). It can be noted as the highest node value is observed over the cross-hub  $c_0^2(3)$ . Finally, Fig. 4 illustrates the average normalized squared error  $\text{NMSE} := \frac{\|\tilde{s}_1^{\ell,m} - s_1^{\ell,m}\|}{\|\tilde{s}_1^{\ell,m}\|}$  versus the signal-to-noise ratio  $\text{SNR} = \sigma_1^2 / \sigma_n^2$ . As expected,

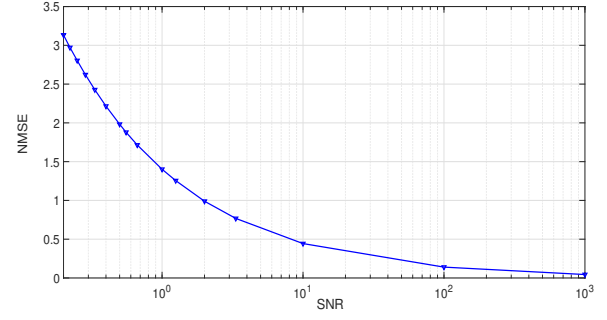


Fig. 4. Normalized mean squared error versus SNR.

the estimation error decreases as the SNR increases.

## VI. CONCLUSIONS

In this paper, we introduced the processing of signals defined on cell multicomplexes, a new representation of topological spaces capable of capturing both intra- and inter-layers higher-order interactions across different networks. We showed how cross-Laplacians matrices are effective algebraic descriptors for representing signals over CMCs. We focused on filtering noisy cross-edges flows, showing how to identify harmonic cross-hubs on one layer that control the functionalities and topology of other layers. Future developments will focus on extending the proposed framework from both a theoretical and an applied perspective.

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