Enhancing Fundamental Frequency Estimation of Multi-Harmonic Signals using Spectral Amplitude Knowledge

Daniel Guger*†§, Oliver Lang*†§, Stefan Schuster‡, Stefan Scheiblhofer‡,
Alexander Haberl‡, Clemens Staudinger‡, Lukas Schiefermüller‡, and Mario Huemer*
*Institute of Signal Processing, Johannes Kepler University Linz, Austria
†Christian Doppler Laboratory for Steel Industry Signal Processing and Machine Learning, Austria
‡voestalpine Stahl GmbH, Linz, Austria
§The authors contributed equally to this work.

***Abstract*—Fundamental frequency estimation of periodic signals constitutes an important topic in the signal processing domain, with a considerable number of applications in audio, sonar, or industrial purposes like condition monitoring, etc. While prominent methods like the maximum likelihood (ML) fundamental frequency estimator offer asymptotically efficient solutions, they neglect certain prior knowledge that may be available. This work considers prior knowledge of the spectral amplitudes that can be used to enhance the ML estimator. We therefore derive a ML fundamental frequency estimator that beneficially utilizes this knowledge. Monte-Carlo simulations emphasize the improved performance of the proposed estimator compared to the classical ML fundamental frequency estimator, for cases where the spectral amplitudes are perfectly or even only approximately known.**

***Index Terms*—Fundamental frequency estimation, CRLB, known spectral amplitudes, maximum likelihood**

I. Introduction

Periodic signals occur, e.g., in audio [1], sonar [2], and in industrial applications like condition monitoring [3], etc. These signals can be modeled as a weighted sum of sinusoids, whose frequencies are integer multiples of a common fundamental frequency ψ_0, and are often called harmonic. A commonly used representation is the multi-harmonic signal model

$$s[n] = \sum_{\ell=1}^{L} A_\ell \cos(2\pi\ell\psi_0 n + \phi_\ell), \quad 0 \leq n \leq N-1, \quad (1)$$

with number of measurements N, known model order L, unknown (normalized) fundamental frequency ψ_0, spectral amplitudes $A_\ell > 0$, and phases $\phi_\ell \in [-\pi, \pi)$.

In many applications, the fundamental frequency of harmonic signals is of particular interest, and numerous fundamental frequency estimators have been developed [4], including non-parametric methods [5]–[8], subspace-based methods [9], [10], and Bayesian methods [11], [12]. Model-based fundamental frequency estimators include the asymptotically efficient maximum likelihood (ML) fundamental frequency estimator [13] and a computationally efficient approximation of it called the approximate nonlinear least squares (ANLS) fundamental frequency estimator [4, p. 42ff.]. Interestingly, the derivations of both estimators have, as an embedded step, the ML estimation of the spectral amplitudes A_ℓ, since they are usually unknown and need to be estimated to derive the ML cost function.

However, in some applications, the spectral amplitudes may not be completely unknown but rather (approximately) known up to a constant scaling factor, i.e., $A_\ell = V\bar{A}_\ell$ with known $\bar{A}_\ell$ and unknown scaling factor V. A prominent case is when the input of a known linear time-invariant (LTI) system is a Dirac-comb-like signal such that the periodic signal $s[n]$ consists of scaled pulses that reflect the system's impulse response. An ML fundamental frequency estimator utilizing this knowledge does not exist in literature to the best of the authors' knowledge. Related literature includes [14], where a quite similar scenario was investigated. The resulting estimators are, however, different. Knowledge that the scaling factor is common to all linear parameters has not been incorporated, as we will show. Another related literature is [12], where the authors mentioned that the correlation structure of the amplitudes of the harmonics can be used as prior information in their Bayesian fundamental frequency estimation framework. A related approach was investigated in [11], where every amplitude was rewritten as a product of a known weight and a time-varying part that needs to be tracked within a Bayesian framework. The weights can be interpreted as prior knowledge on the spectral components and are chosen such that the time-varying parts are on a similar scale. In [15], [16], it was shown that the asymptotic ($N \gg 1/(2\pi\psi_0)$) Cramér-Rao lower bound (CRLB) for the case of additive white Gaussian noise (AWGN) is independent of whether the spectral amplitudes A_ℓ are known or not. The asymptotic CRLB for both cases is

$$\mathrm{var}(\hat{\psi}_0) \geq \frac{1}{(2\pi)^2}\frac{12}{N^3\mathrm{PSNR}}, \quad (2)$$

with the pseudo signal-to-noise ratio (PSNR) defined as $\mathrm{PSNR} = \sum_{\ell=1}^{L}(\ell^2 A_\ell^2)/(2\sigma^2)$. In [17], an estimator similar to the ANLS fundamental frequency estimator is used on audio signals. This estimator weights the spectral components according to $0.84^{\ell-1}$ to account for the tendency of higher-order harmonics having less power in speech signals than lower-order harmonics. In [1], the knowledge of the smoothness of the spectral amplitudes is utilized to separate two speech signals with overlapping spectral components. In [18], a rather ad-hoc estimation procedure incorporating smoothing and splitting of the time-series has been proposed.

In this work, we consider the case where the spectral amplitudes follow $A_\ell = V\bar{A}_\ell$ with known $\bar{A}_\ell$. We first derive the ML fundamental frequency estimator under this assumption, and propose a computationally efficient modified ANLS fundamental frequency estimator that can be efficiently implemented using the fast Fourier transformation (FFT). It will turn out that the proposed modified ANLS fundamental frequency estimator performs a weighting similar to the approach in [17], but with known scaled spectral amplitudes $\bar{A}_\ell$.

The proposed fundamental frequency estimators incorporating $\bar{A}_\ell$ achieve better mean squared error (MSE) performances closer to the CRLB in the moderate- and low-signal-to-noise ratio (SNR) region. Moreover, they feature a better threshold SNR value [19], [20] compared to all competitive estimators.

II. Standard ML and ANLS Estimators

In the following, the standard estimators without knowledge of A_ℓ are revisited, which will serve as a performance benchmark later on. The proposed estimators incorporating knowledge of A_ℓ are derived in the follow-up section.

The real-valued, multi-harmonic signal in (1) is considered. Using $\alpha_\ell = A_\ell \cos(\phi_\ell)$ and $\beta_\ell = -A_\ell \sin(\phi_\ell)$, this model can be expanded to

$$s[n] = \sum_{\ell=1}^{L} \alpha_\ell \cos(2\pi\ell\psi_0 n) + \beta_\ell \sin(2\pi\ell\psi_0 n), \qquad (3)$$

such that the fundamental frequency ψ_0 is the only nonlinear parameter in the signal model. We assume N measurements are given by

$$x[n] = s[n] + w[n] \qquad (4)$$

with $w[n] \sim \mathcal{N}(0, \sigma^2)$ being AWGN.

Assembling all linear parameters α_ℓ and β_ℓ to vectors $\boldsymbol{\alpha}^T = [\alpha_1, \ldots, \alpha_L]$ and $\boldsymbol{\beta}^T = [\beta_1, \ldots, \beta_L]$, then the measurement vector $\mathbf{x}^T = [x[0], \ldots, x[N-1]]$ can be described as

$$\mathbf{x} = \mathbf{H}(\psi_0)\boldsymbol{\eta} + \mathbf{w} \qquad (5)$$

with $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}_N)$, $\mathbf{I}_N$ being the identity matrix of size $N \times N$, the linear parameters $\boldsymbol{\eta}^T = \begin{bmatrix}\boldsymbol{\alpha}^T & \boldsymbol{\beta}^T\end{bmatrix}$, and with

$$\mathbf{H}^T(\psi_0) = \begin{bmatrix} \cos(2\pi 1\psi_0 0) & \ldots & \cos(2\pi 1\psi_0(N-1)) \\ \vdots & \vdots & \vdots \\ \cos(2\pi L\psi_0 0) & \ldots & \cos(2\pi L\psi_0(N-1)) \\ \sin(2\pi 1\psi_0 0) & \ldots & \sin(2\pi 1\psi_0(N-1)) \\ \vdots & \vdots & \vdots \\ \sin(2\pi L\psi_0 0) & \ldots & \sin(2\pi L\psi_0(N-1)) \end{bmatrix}. \qquad (6)$$

For the considered case of AWGN with equal noise variances, the position of the maximum of the likelihood function coincides with the position of the minimum of the least squares (LS) cost function [13], [21, p. 71ff.] and is given by

$$\hat{\psi}_0 = \arg\min_{\psi_0} (\mathbf{x} - \mathbf{H}(\psi_0)\boldsymbol{\eta})^T (\mathbf{x} - \mathbf{H}(\psi_0)\boldsymbol{\eta}). \qquad (7)$$

Solving it as a separable LS problem [22, p. 254ff.], the linear parameters are estimated by

$$\hat{\boldsymbol{\eta}} = \left(\mathbf{H}^T(\psi_0)\mathbf{H}(\psi_0)\right)^{-1} \mathbf{H}^T(\psi_0)\mathbf{x}, \qquad (8)$$

which can be inserted in (7), yielding the final ML estimator for ψ_0 without knowledge of the spectral amplitudes A_ℓ

$$\hat{\psi}_0 = \arg\max_{\psi_0} \mathbf{x}^T\mathbf{H}(\psi_0)\left(\mathbf{H}^T(\psi_0)\mathbf{H}(\psi_0)\right)^{-1}\mathbf{H}^T(\psi_0)\mathbf{x}. \qquad (9)$$

The estimator's cost function is usually maximized using a grid search.

A simplification can be made if $1/(2N) \ll \psi_0$ [23, Problem 4.21], and if $L\psi_0 \ll 1/2$. Then, the matrix inversion can be approximated by $\left(\mathbf{H}^T(\psi_0)\mathbf{H}(\psi_0)\right)^{-1} \approx 2/N\mathbf{I}_N$ [21, Algorithm 12.3]. Hence, an approximate ML fundamental frequency estimator, the ANLS for unknown spectral amplitudes A_ℓ can be obtained according to

$$\hat{\psi}_0 = \arg\max_{\psi_0} \frac{2}{N} \sum_{\ell=1}^{L} |X(\ell\psi_0)|^2, \qquad (10)$$

where $X(\psi)$ is the windowed discrete-time Fourier transformation (DTFT) of $x[n]$ given by

$$X(\psi) = \sum_{n=0}^{N-1} x[n] \exp(-\mathrm{j}2\pi\psi n). \qquad (11)$$

The windowed DTFT can be efficiently evaluated using the FFT [24]. The corresponding CRLB is given by (2).

III. Derivation of the Proposed Estimators

As already mentioned, assuming the spectral amplitudes A_ℓ were completely known leads to the same asymptotic CRLB as in (2) [15]. The asymptotic ($N \to \infty$) performance of the ML estimator can therefore not be improved by incorporating knowledge of the spectral amplitudes. However, it will be shown that the performance in low- and moderate-SNR regions as well as the threshold SNR value benefit from such knowledge.

In the following, we assume that the spectral amplitudes are known up to a constant scaling factor $A_\ell = V\bar{A}_\ell$ with known $\bar{A}_\ell > 0$ and unknown scaling factor $V > 0$. This leads to the model

$$x[n] = V\sum_{\ell=1}^{L} \bar{A}_\ell \cos(2\pi\ell\psi_0 n + \phi_\ell) + w[n], \qquad (12)$$

where $w[n] \sim \mathcal{N}(0, \sigma^2)$ is AWGN.

Let's fix ψ_0 for a moment. We will first discuss estimators for ϕ_ℓ and V for a given ψ_0, and will later on find ψ_0 via a grid search.

Assuming that the cosines in (12) are approximately orthogonal for different values of ℓ, (i.e.,

$$\mathbf{H}(\psi_0) = \begin{bmatrix} \cos(2\pi 1\psi_0 0 + \hat{\phi}_1) & \cos(2\pi 2\psi_0 0 + \hat{\phi}_2) & \dots & \cos(2\pi L\psi_0 0 + \hat{\phi}_L) \\ \cos(2\pi 1\psi_0 1 + \hat{\phi}_1) & \cos(2\pi 2\psi_0 1 + \hat{\phi}_2) & \dots & \cos(2\pi L\psi_0 1 + \hat{\phi}_L) \\ \vdots & \vdots & & \vdots \\ \cos(2\pi 1\psi_0 (N-1) + \hat{\phi}_1) & \cos(2\pi 2\psi_0 (N-1) + \hat{\phi}_2) & \dots & \cos(2\pi L\psi_0 (N-1) + \hat{\phi}_L) \end{bmatrix} \quad (19)$$

$\left(\mathbf{H}^T(\psi_0)\mathbf{H}(\psi_0)\right)^{-1} \approx 2/N\mathbf{I}_N$), one can show that the approximate ML estimator for ϕ_ℓ is given by

$$\hat{\phi}_\ell = -\operatorname{atan2}\left(\sum_{n=0}^{N-1} x[n]\sin(2\pi\ell\psi_0 n), \sum_{n=0}^{N-1} x[n]\cos(2\pi\ell\psi_0 n)\right). \quad (13)$$

The proof is similar to [22, p. 167ff.] and is omitted in this work. We note that a symmetric sampling interval [25], [26] may improve the estimation performance of $\hat{\phi}_\ell$, however, for the sake of simplicity, we continue with (13).

Next, the phases ϕ_ℓ in (12) are replaced by their estimates in (13), leading to the vector notation

$$\mathbf{x} = V\mathbf{h}(\psi_0) + \mathbf{w}, \quad (14)$$

with $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}_N)$ and

$$\mathbf{h}(\psi_0) = \begin{bmatrix} \sum_{\ell=1}^{L} \bar{A}_\ell \cos(2\pi\ell\psi_0 0 + \hat{\phi}_\ell) \\ \sum_{\ell=1}^{L} \bar{A}_\ell \cos(2\pi\ell\psi_0 1 + \hat{\phi}_\ell) \\ \sum_{\ell=1}^{L} \bar{A}_\ell \cos(2\pi\ell\psi_0 2 + \hat{\phi}_\ell) \\ \vdots \\ \sum_{\ell=1}^{L} \bar{A}_\ell \cos(2\pi\ell\psi_0 (N-1) + \hat{\phi}_\ell) \end{bmatrix}. \quad (15)$$

Then, the approximate ML estimator for V is given by

$$\hat{V} = \left(\mathbf{h}^T(\psi_0)\mathbf{h}(\psi_0)\right)^{-1} \mathbf{h}^T(\psi_0)\mathbf{x}. \quad (16)$$

Inserting this result into the model in (14), and using the log-likelihood function for performing a grid search for estimating ψ_0 leads to

$$\hat{\psi}_0 = \arg\max_{\psi_0} \mathbf{x}^T\mathbf{h}(\psi_0)\left(\mathbf{h}^T(\psi_0)\mathbf{h}(\psi_0)\right)^{-1}\mathbf{h}^T(\psi_0)\mathbf{x}. \quad (17)$$

Due to several assumptions made during the derivation, this result is only an approximate ML estimator for the fundamental frequency ψ_0. However, we refer to this estimator simply as the ML estimator in the remainder of this work.

Further simplifications can be made when rewriting the vector $\mathbf{h}(\psi_0)$ as

$$\mathbf{h}(\psi_0) = \mathbf{H}(\psi_0)\mathbf{a}, \quad (18)$$

where $\mathbf{H}(\psi_0)$ is defined according to (19) and $\mathbf{a}^T = \begin{bmatrix}\bar{A}_1 & \bar{A}_2 & \dots & \bar{A}_L\end{bmatrix}$. Combining (18) with the ML estimator in (17) produces

$$\hat{\psi}_0 = \arg\max_{\psi_0} \mathbf{x}^T\mathbf{H}(\psi_0)\mathbf{a}\left(\mathbf{a}^T\mathbf{H}^T(\psi_0)\mathbf{H}(\psi_0)\mathbf{a}\right)^{-1}\mathbf{a}^T\mathbf{H}^T(\psi_0)\mathbf{x}. \quad (20)$$

Since $\mathbf{H}^T(\psi_0)\mathbf{H}(\psi_0)$ is approximately a scaled identity matrix, one obtains

$$\hat{\psi}_0 \approx \arg\max_{\psi_0} \mathbf{x}^T\mathbf{H}(\psi_0)\mathbf{a}\left(\mathbf{a}^T\mathbf{a}\right)^{-1}\mathbf{a}^T\mathbf{H}^T(\psi_0)\mathbf{x}. \quad (21)$$

The term $\mathbf{a}^T\mathbf{a}$ is a scalar that does not depend on ψ_0. Dismissing it leads to

$$\hat{\psi}_0 \approx \arg\max_{\psi_0} \mathbf{x}^T\mathbf{H}(\psi_0)\mathbf{a}\mathbf{a}^T\mathbf{H}^T(\psi_0)\mathbf{x} \quad (22)$$

$$\approx \arg\max_{\psi_0} |\mathbf{a}^T\mathbf{H}^T(\psi_0)\mathbf{x}|^2. \quad (23)$$

The squaring operation does not change the position of the maximum and can be dismissed. Moreover, it can be shown that if $\mathbf{H}^T(\psi_0)\mathbf{x}$ is evaluated at the true fundamental frequency, for high enough SNR, and if the estimates $\hat{\phi}_\ell$ are close to the true phases ϕ_ℓ, the term $\mathbf{H}^T(\psi_0)\mathbf{x}$ would approximately coincide with the magnitude of the windowed DTFT $|X(\psi)|$ evaluated at the frequencies $\psi = \ell\psi_0$ with $\ell = 1, \dots, L$. Using the windowed DTFT instead due to its simpler computation, and replacing the vector notation by a summation results in

$$\hat{\psi}_0 \approx \arg\max_{\psi_0} \left|\sum_{\ell=1}^{L} \bar{A}_\ell\,|X(\ell\psi_0)|\right|. \quad (24)$$

Since both arguments in the sum are real-valued and positive, the outermost magnitude operator can be dismissed, leading to the final ANLS for known spectral amplitudes $\bar{A}_\ell$ is finally given by

$$\hat{\psi}_0 = \arg\max_{\psi_0} \sum_{\ell=1}^{L} \bar{A}_\ell |X(\ell\psi_0)|. \quad (25)$$

This estimator weights the spectral signal components $|X(\ell\psi_0)|$ with the known spectral amplitudes $\bar{A}_\ell$.

It is also interesting to compare this result to a result in [27] for the case of weak signal in colored noise. There a similar weighting is performed with the coefficients of a prewhitener based on estimated autoregressive (AR) noise process parameters.

IV. Simulation results

In the following, Monte-Carlo simulations are presented for the ML fundamental frequency estimators and the ANLS fundamental frequency estimators for both cases, unknown (UA) and known spectral amplitudes $\bar{A}_\ell$ (KA). Moreover, the performance degradation when $\bar{A}_\ell$ are subject to an error (AA) is empirically analyzed.

Our harmonic test signal is realized as a periodic repetition of Hann windows

$$w_\mathrm{H}[n] = \frac{1}{2}\left(1 - \cos\left(\frac{2\pi n}{H}\right)\right)\mathrm{rect}_H[n] \quad (26)$$

of length H, where the rectangular window is given by

$$\mathrm{rect}_H[n] = \begin{cases} 1, & 0 \leq n \leq H-1, \\ 0, & \text{otherwise.} \end{cases} \quad (27)$$

The so-called Hann-comb of frequency $\psi_0 = 1/P$ follows as

$$s_{\mathrm{H}}[n] = w_{\mathrm{H}}[n] * \sum_{p=-\infty}^{\infty} \delta[n - pP], \quad 0 \leq n \leq N-1, \quad (28)$$

where $\delta[n]$ is the discrete Dirac delta function, and where P is the integer-valued period. We choose a signal length of $N = 2^{11} = 2048$ and a pulse width of $H = 10$ samples. For this signal model, a closed-form expression for the complex spectral amplitudes $\underline{A}_\ell$ can be shown to be

$$\underline{A}_\ell = \frac{1}{P}\left[\frac{1}{\sin\left(\frac{\pi\ell}{P}\right)} - \frac{1}{2}\left(\frac{\exp\left(\mathrm{j}\frac{\pi}{H}\right)}{\sin\left(\frac{\pi\ell}{P} + \frac{\pi}{H}\right)} + \frac{\exp\left(-\mathrm{j}\frac{\pi}{H}\right)}{\sin\left(\frac{\pi\ell}{P} - \frac{\pi}{H}\right)}\right)\right] \cdot \sin\left(\frac{\pi H \ell}{P}\right) \exp\left(-\mathrm{j}\frac{\pi\ell(H-1)}{P}\right). \quad (29)$$

The spectral amplitudes A_ℓ then follow as the absolute value of their complex counterparts in form of $A_\ell = |\underline{A}_\ell|$. For efficient computation of the windowed DTFT using the FFT, the measurement signal $x[n]$ is zero-padded to a length of 2^{17} samples. For each trial, the fundamental frequency ψ_0 was sampled from a uniform distribution $\psi_0 \in \mathcal{U}(1.94 \cdot 10^{-2}, 3.06 \cdot 10^{-2})$, and then rounded such that the period $P = 1/\psi_0$ is integer-valued. Furthermore, the grid search in all simulations is limited to a search range $\hat{\psi}_0 \in \Psi = [\psi_{\min}, \psi_{\max}] = [1.81 \cdot 10^{-2}, 3.19 \cdot 10^{-2}]$, chosen such that fractions or integer multiples of any frequency within Ψ are excluded, i.e., $2\psi_{\min} > \psi_{\max}$.

Fig. 1 compares the resulting MSE performances. Compared to the ML and ANLS fundamental frequency estimators for unknown spectral amplitudes, the proposed estimators are closer to the CRLB and feature better MSE performances by approximately a factor of 2.3 measured at a PSNR of -3 dB, exemplarily. Moreover, the proposed estimators offer a threshold PSNR improvement by about 1 dB.

Strictly speaking, the spectral amplitudes in (29) depend on $\psi_0 = 1/P$ and have to be evaluated for each point of the grid search. Fortunately, for moderate variations of the fundamental frequency ψ_0, the spectral amplitudes A_ℓ only vary in a limited manner as shown in Fig. 2. The spectral amplitudes of the first 20 harmonics $\ell \in \{1, \ldots, 20\}$ are illustrated when varying the fundamental frequency ψ_0 by about ± 10 %. One can see that the spectral amplitudes exhibit a quite similar shape despite the different fundamental frequencies.

The proposed estimators were employed with erroneous spectral amplitudes similar to those shown in Fig. 2. This was done by choosing $\bar{A}_\ell = A_\ell|_{\psi=\tilde{\psi}}$ with $\tilde{\psi}$ randomly chosen from $\tilde{\psi}_0 \in \mathcal{U}(0.9\psi_0, 1.1\psi_0)$. The resulting MSE performances are indicated by the dashed curves in Fig. 1. As can be seen, almost no loss in MSE performance compared to the estimators having information about the true spectral amplitudes is observed.

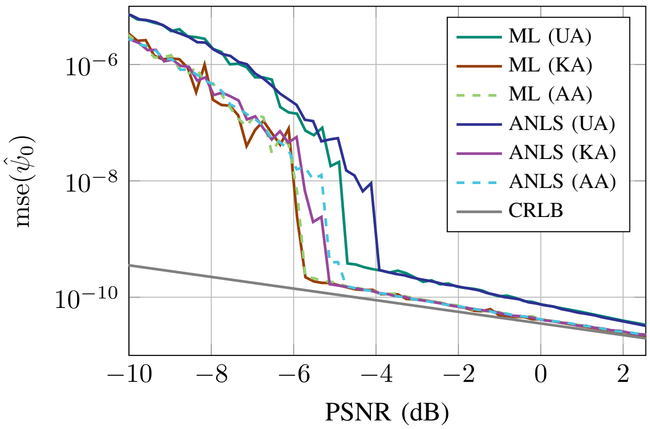

Fig. 1. Performance comparison of the ML and the ANLS fundamental frequency estimators for unknown (UA), known (KA) and approximately known spectral amplitudes A_ℓ (AA).

The preservation of the improved performance concerning both, lower threshold value and MSE performance closer to the CRLB, even when only approximate knowledge of the spectral amplitudes $\bar{A}_\ell$ is available, indicates the robustness of the proposed estimators and increases their usability.

V. Conclusion

In this work, the task of fundamental frequency estimation of a multi-harmonic (periodic) signal was investigated for the special case when the spectral amplitudes of the harmonic components are known up to a constant scaling factor. The ML fundamental frequency estimator that incorporates this knowledge as well as a computationally efficient approximation called the ANLS fundamental frequency estimator were derived. Both proposed estimators were compared to their counterparts that do not incorporate the information about the spectral amplitudes. This comparison showed that, although all estimators approach the same CRLB asymptotically, the

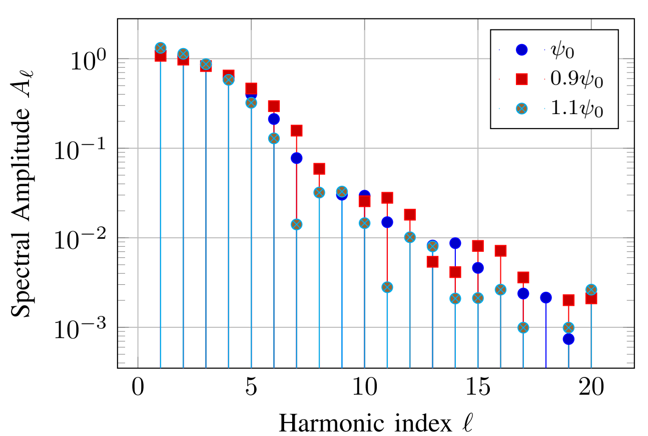

Fig. 2. Spectral amplitudes A_ℓ of the first $\ell \in \{1, \ldots, 20\}$ harmonics of the Hann-comb signal from (28) for different fundamental frequencies $\psi_0 \in \{2.22 \cdot 10^{-2}, 2.5 \cdot 10^{-2}, 2.77 \cdot 10^{-2}\}$.

proposed estimators outperform their competitors in the low- and moderate-PSNR region and offer a lower threshold value. Finally, the case where the spectral amplitudes are subject to an error was analyzed, indicating that for moderate errors no severe performance degradation is observed.

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