Robust Semiparametric Time-Delay and Doppler Estimation: Analysis of R- and M-Estimators

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***Abstract*—This paper investigates time-delay and Doppler estimation in the presence of unknown heavy-tailed disturbances. Traditional approaches, such as the maximum likelihood estimator, achieve optimal mean squared error performance under the unrealistic assumption of perfect prior knowledge of the noise distribution. To address this limitation, previous work introduced a *rank-based* and *distribution-free* R-estimator, which is shown to be *parametrically efficient*, attaining the *classical* Cramér-Rao Bound irrespective of the unknown noise distribution, provided it belongs to the family of Complex Elliptically Symmetric distributions. The aim of this paper is to analyse and compare the performance of the R-estimator with an M-estimator, a widely used robust estimation approach. Specifically, we analyse their statistical efficiency for the time-delay and Doppler estimation problem, under various noise conditions. Furthermore, we propose to combine both estimators, leveraging their complementary strengths to enhance estimation performance. Numerical simulations illustrate the benefits of this hybrid approach.**

***Index Terms*—Complex elliptically symetric distribution, Semiparametric Cramér-Rao bound, R-estimators, time-delay and Doppler estimation, band-limited signals.**

I. Introduction

Time-delay and Doppler estimation are critical tasks in numerous engineering applications, including communications, radar, and navigation systems [1]–[6]. These parameters play a fundamental role in signal synchronization, target tracking, and geolocation, making their accurate estimation essential for system performance. A key aspect of this problem is determining the best achievable accuracy, typically measured in terms of mean squared error (MSE). Under standard parametric assumptions, the Cramér-Rao Bound (CRB) [7], [8] provides a theoretical lower bound on the variance of any unbiased estimator. Moreover, it is well known that the Maximum Likelihood Estimator (MLE) asymptotically attains the CRB under certain regularity conditions [9]. Consequently, extensive research has been conducted to derive CRB formulations for time-delay and Doppler estimation across various signal models, spanning both narrow-band and wide-band signals [2], [10]–[18]. In real-world applications, however, the assumed signal model at the receiver may deviate from the actual one. Several recent studies have investigated such mismatches. In [19]–[22], mismatches arise due to multipath, interference, or high receiver dynamics. Furthermore, in [23], [24], the impact of non-Gaussian, heavy-tailed Complex Elliptically Symmetric (CES) noise distributions has been analyzed, where the assumed model is a standard complex normal distribution. These studies adopt the theory of model misspecification [25], [26], where estimation performance is characterized in terms of pseudo-true parameters and the Misspecified CRB (MCRB). Specifically, the MCRB provides a lower bound to the error covariance of the Misspecified MLE (MMLE) when the assumed model does not match the true distribution [25, Theo. 2], [26, Sec. 4.4.3]. Notably, even when the noise follows a heavy-tailed CES distribution, the Gaussian-based MMLE remains $\sqrt{N}$-consistent with respect to the true parameters as the number of observations increases.

Despite its consistency, the MMLE does not guarantee asymptotic efficiency under unknown noise distributions. To address this issue, semiparametric estimation theory provides a rigorous framework for designing robust yet efficient estimators, as recently explored in [27]–[30] for CES distributions. The semiparametric framework allows for the derivation of a lower bound known as the Semiparametric CRB (SCRB), which represents the lowest achievable MSE for any consistent estimator under an unspecified CES distribution. Crucially, for time-delay and Doppler estimation, it can be shown that the SCRB coincides with the CRB of the true distribution [28]. This implies that a semiparametric efficient estimator will also be parametrically efficient.

In our previous work [31], a *rank-based*, *distribution-free* R-estimator was proposed, achieving semiparametric efficiency, i.e., attaining the CRB regardless of the unknown CES noise distribution. In this study, we extend this analysis by comparing the performance of the R-estimator with an M-estimator, a widely used robust alternative that is less sensitive to heavy-tailed disturbances. We examine their statistical perfomance under unknown non-Gaussian noise conditions. Furthermore, we propose to combine both estimators, leveraging their respective advantages to enhance estimation performance.

II. Signal model

We consider a system where a band-limited signal $s(t)$, with bandwidth B, is transmitted over a carrier frequency f_c ($\lambda_c = c/f_c$, $\omega_c = 2\pi f_c$) from a transmitter T at position $\boldsymbol{p}_T(t)$ to a receiver R at position $\boldsymbol{p}_R(t)$. Assuming a first-order approximation, the transmitted distance is given by

$$p_{TR} \approx c(\bar{\tau} + \bar{b}t), \tag{1}$$

where $\bar{\tau} = \frac{\|\boldsymbol{p}_T(0)-\boldsymbol{p}_R(0)\|}{c}$ and $\bar{b} = \frac{\|\boldsymbol{v}\|}{c}$, with $\boldsymbol{v}$ denoting the relative velocity vector between the transmitter and the receiver. Under the narrowband assumption, the received signal after baseband demodulation can be expressed as [10], [16]

$$x\left(t;\bar{\boldsymbol{\eta}}\right) = \bar{\alpha} s\left(t-\bar{\tau}\right) e^{-j2\pi f_c\left(\bar{b}\left(t-\bar{\tau}\right)\right)} + n\left(t\right), \quad (2)$$

where $\bar{\boldsymbol{\eta}} = \left(\bar{\tau}, \bar{b}\right)^T$ and $\bar{\alpha}$ is a complex gain. The discrete vector signal model is constructed from $N = N_1 - N_2 + 1$ samples taken at $T_s = 1/F_s = 1/B$:

$$\mathbf{x} = \bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) + \mathbf{n}, \quad (3)$$

where $\mathbf{x} = \left(\ldots, x\left(kT_s\right), \ldots\right)^\top$, $\mathbf{n} = \left(\ldots, n\left(kT_s\right), \ldots\right)^\top$ represents the noise samples, and $N_1 \leq k \leq N_2$. The noise samples $n(kT_s)$ are assumed to be independent and identically distributed (i.i.d.) following a CES distribution, i.e., $n(kT_s) \sim CES(0, \bar{\sigma}_n^2, \bar{g})$, where $\bar{\sigma}_n^2$ is the unknown noise power, and $\bar{g}$ is an unspecified density generator [32]. The signal component is given by

$$\boldsymbol{\mu}(\bar{\boldsymbol{\eta}}) = (\ldots, s(kT_s - \bar{\tau}) e^{-j2\pi f_c(\bar{b}(kT_s-\bar{\tau})}, \ldots)^\top. \quad (4)$$

The unknown deterministic parameters are collected in the vector $\bar{\boldsymbol{\epsilon}}^\top = (\bar{\sigma}_n^2, \bar{\rho}, \bar{\Phi}, \bar{\boldsymbol{\eta}}^\top) = (\bar{\sigma}_n^2, \bar{\boldsymbol{\theta}}^\top)$, where $\bar{\alpha} = \bar{\rho} e^{j\bar{\Phi}}$ with $\bar{\rho} \in \mathbb{R}^+$ and $0 \leq \bar{\Phi} \leq 2\pi$. The underlying data-generating model is then characterized by the probability density function

$$p_{\bar{\boldsymbol{\epsilon}}}(\mathbf{x}; \bar{\boldsymbol{\epsilon}}) = \Pi_{k=N_1}^{N_2} p_{\bar{\boldsymbol{\epsilon}}}(x_k, \bar{\boldsymbol{\epsilon}}), \quad (5)$$

with

$$p_{\bar{\boldsymbol{\epsilon}}}(x_k, \bar{\boldsymbol{\epsilon}}) = CES(\bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}), \bar{\sigma}_n^2, \bar{g}). \quad (6)$$

Since $\bar{g}$ is left unspecified, the likelihood function is unknown, preventing the derivation of a MLE for $\bar{\boldsymbol{\epsilon}}$. For further reference, we recall that, according to the Stochastic Representation Theorem [32, Theo. 3], the received signal can be rewritten as

$$x_k =_d \bar{\alpha}\boldsymbol{\mu}_k(\bar{\boldsymbol{\eta}}) + \sqrt{Q_k}\bar{\sigma}_n u_k =_d f_k(\bar{\boldsymbol{\theta}}) + \sqrt{\mathcal{Q}}\bar{\sigma}_n u_k, \quad (7)$$

where u_k is a complex univariate random variable uniformly distributed on $\mathbb{C}S \triangleq \{u \in \mathbb{C} | |u| = 1\}$, i.e., $u_k \sim U(\mathbb{C}S)$. Moreover,

$$Q_k \triangleq |x_k - f_k(\bar{\boldsymbol{\theta}})|^2 / \bar{\sigma}_n^2 =_d \mathcal{Q} \quad (8)$$

is a positive random variable independent of u_k, with probability density function $p_{\mathcal{Q}}(q) = \delta_g^{-1}\bar{g}(q)$, where $\delta_g \triangleq \int_0^\infty \bar{g}(q)dq$ is a normalization constant (see [32, Eq. (19)]). To address the scale ambiguity between $\bar{\sigma}_n^2$ and $\bar{g}$, we impose the constraint $E\{\mathcal{Q}\} = 1$, which allows $\bar{\sigma}_n^2$ to be interpreted as the *statistical power* P of the received data x_k [32, Sec. III.C].

A. Closed-form expression of the SCRB for $\bar{\boldsymbol{\theta}}$

In accordance with semiparametric theory [33], the absence of prior knowledge regarding the density generator $\bar{g}$ can be incorporated into the derivation of the semiparametric Cramér-Rao Bound (SCRB) for $\bar{\boldsymbol{\theta}}$ by treating $\bar{g}$ as a *functional* nuisance parameter. Specifically, the SCRB is defined as the inverse of the Semiparametric Efficient Fisher Information Matrix (SFIM). The explicit computation of the SFIM for CES-distributed data can be achieved using the Semiparametric Slepian-Bangs formula, which has been established under broad conditions in [28, eq.(47)]. For our particular setting, it can be shown that the SFIM coincides with the Fisher Information Matrix (FIM) of the true distribution. Consequently, we obtain a closed-form expression given by:

$$\mathbf{I}(\bar{\boldsymbol{\theta}}) = \frac{2E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}}{\bar{\sigma}_n^2} \Re\left\{\left(\frac{\partial\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})}{\partial\boldsymbol{\theta}}\right)^H \left(\frac{\partial\bar{\alpha}\boldsymbol{\mu}(\bar{\boldsymbol{\eta}})}{\partial\boldsymbol{\theta}}\right)\right\}, \quad (9)$$

where $\bar{\psi}(t) \triangleq d\ln\bar{g}(t)/dt$, $\mathcal{Q}$ is as defined in (7), and the expectation is taken with respect to its density $p_{\mathcal{Q}}$. Given that the SFIM matches the FIM, it follows directly that:

$$\mathrm{SCRB}(\bar{\boldsymbol{\theta}}|\bar{g}) = \mathbf{I}(\bar{\boldsymbol{\theta}})^{-1} = \mathrm{CRB}(\bar{\boldsymbol{\theta}}) \; \forall \bar{g}. \quad (10)$$

To conclude this section, we note that a closed-form expression for $\mathbf{I}(\bar{\boldsymbol{\theta}}|\bar{g})$ can be obtained as follows [16]:

$$\mathbf{I}(\bar{\boldsymbol{\theta}}) = E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}\mathbf{K}(\bar{\boldsymbol{\theta}}), \quad \mathbf{K}(\bar{\boldsymbol{\theta}}) \triangleq \frac{2F_s}{\bar{\sigma}_n^2}\Re\left\{\mathbf{Q}\mathbf{W}\mathbf{Q}^H\right\} \quad (11)$$

where $\mathbf{W}$ is defined as:

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2^* & w_3^* \\ w_2 & w_{2,2} & w_4^* \\ w_3 & w_4 & w_{3,3} \end{bmatrix}. \quad (12)$$

The elements of $\mathbf{W}$ are expressed in terms of the baseband signal samples as follows:

$$w_1 = \frac{1}{F_s}\mathbf{s}^H\mathbf{s}, \quad w_2 = \frac{1}{F_s^2}\mathbf{s}^H\mathbf{D}\mathbf{s}, \quad w_3 = \mathbf{s}^H\boldsymbol{\Lambda}\mathbf{s},$$
$$w_4 = \frac{1}{F_s}\mathbf{s}^H\mathbf{D}\boldsymbol{\Lambda}\mathbf{s}, \quad w_{2,2} = \frac{1}{F_s^3}\mathbf{s}^H\mathbf{D}^2\mathbf{s}, \quad w_{3,3} = F_s\mathbf{s}^H\mathbf{V}\mathbf{s}.$$

The matrix $\mathbf{Q}$ is given by:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ j\bar{\rho} & 0 & 0 \\ j\bar{\rho}2\pi f_c\bar{b} & 0 & -\bar{\rho} \\ 0 & -j\bar{\rho}2\pi f_c & 0 \end{bmatrix}. \quad (13)$$

where $\mathbf{s} = (\ldots, s(kT_s), \ldots)_{N_1 \leq k \leq N_2}^\top$ represents the baseband signal samples, and $\mathbf{D}$ is defined as:

$$\mathbf{D} = \mathrm{diag}\left(\ldots, k, \ldots\right)_{N_1 \leq k \leq N_2}. \quad (14)$$

Finally, the matrices $\boldsymbol{\Lambda}$ and $\mathbf{V}$ are defined element-wise as:

$$(\boldsymbol{\Lambda})_{k,k'} = \begin{cases} \frac{(-1)^{|k-k'|}}{k-k'} & \text{if } k' \neq k, \\ 0 & \text{if } k' = k. \end{cases} \quad (15)$$

$$(\mathbf{V})_{k,k'} = \begin{cases} (-1)^{|k-k'|}\frac{2}{(k-k')^2} & \text{if } k' \neq k, \\ \frac{\pi^2}{3} & \text{if } k' = k. \end{cases} \quad (16)$$

III. Efficient R-estimator for $\bar{\boldsymbol{\theta}}$

The R-estimator for $\bar{\boldsymbol{\theta}}$ has been already proposed in [31]:

$$\hat{\boldsymbol{\theta}}_R = \boldsymbol{\theta}^\star + (\sqrt{N}\hat{\alpha})^{-1}[\mathbf{K}(\boldsymbol{\theta}^\star)]^{-1}\widetilde{\Delta}_N(\boldsymbol{\theta}_n^\star). \tag{17}$$

where $\boldsymbol{\theta}^\star$ is a preliminary estimator, i.e. a $\sqrt{N}-$consistent, but not necessarily efficient, estimator of $\bar{\boldsymbol{\theta}}$ and $\mathbf{K}(\boldsymbol{\theta}^\star)$ can be simply calculated directly by substituting the estimates of the preliminary estimator $\boldsymbol{\theta}^\star$ in equation (11). To compute $\hat{\alpha}$ and $\widetilde{\Delta}_N(\boldsymbol{\theta}_n^\star)$, we based on rank theory. Ranks are important in robust statistics due to their *distribution-free* property. Following the general approach discussed in [29], [34], we introduce the following quantities:

$$Q_k^\star \triangleq |x_k - f_k(\boldsymbol{\theta}^\star)|^2/(\sigma_n^\star)^2, \tag{18}$$

$$u_k^\star \triangleq (x_k - f_k(\boldsymbol{\theta}^\star))\,/\left(\sigma_n^\star\sqrt{Q_k^\star}\right). \tag{19}$$

We now define the *ranks* $\{r_k^\star\}_{k=N_1}^{N_2}$ of the (continuous) real random variables $\{Q_k^\star\}_{k=N_1}^{N_2}$ as their position index after having ordered them in an ascending way the ordered statistics [35, Ch. 13]. Using this, we can provide a rank-based approximation for $\Delta_N(\boldsymbol{\theta}^\star)$ and $\hat{\alpha}$. In fact, for a given *score function* $M(\cdot)$, [1] we have:

$$\widetilde{\Delta}_N(\boldsymbol{\theta}^\star) \triangleq \frac{-2}{\sqrt{N}\sigma_n^\star}\sum_{k=N_1}^{N_2} M\left(\frac{r_k}{N+1}\right)\Re\left[(u_k^\star)^*\nabla_{\boldsymbol{\theta}} f_k(\boldsymbol{\theta}^\star)\right]. \tag{20}$$

whose closed-form has been derived in [31] as:

$$\widetilde{\Delta}_N(\boldsymbol{\theta}^\star) = (2F_s)/(\sqrt{N}\sigma^\star)\Re\left\{e^{j\Phi^\star}\mathbf{Q}^\star\boldsymbol{w}_e\right\}. \tag{21}$$

with $\mathbf{Q}^\star$ computed by substituting the estimates of the preliminary estimator $\boldsymbol{\theta}^\star$ in equation (13). Moreover, $\boldsymbol{w}_e = (w_{e_1}, w_{e_2}, w_{e_3})^T$which each elements derived in [31],

$$w_{e_1} = \frac{1}{F_s}\boldsymbol{\mathcal{U}}^H\boldsymbol{V}^{\Delta,0}\left(\frac{\tau^\star}{T_s}\right)\boldsymbol{U}\left(\frac{f_c b^\star}{F_s}\right)\boldsymbol{s},$$
$$w_{e_2} = \frac{1}{F_s^2}\boldsymbol{\mathcal{U}}^H\boldsymbol{V}^{\Delta,0}\left(\frac{\tau^\star}{T_s}\right)\boldsymbol{U}\left(\frac{f_c b^\star}{F_s}\right)\boldsymbol{D}\mathbf{s},$$
$$w_{e_3} = \boldsymbol{\mathcal{U}}^H\boldsymbol{V}^{\Delta,1}\left(\frac{\tau^\star}{T_s}\right)\boldsymbol{U}\left(\frac{f_c b^\star}{F_s}\right)\mathbf{s} + jw_c b^\star w_{e_1},$$

with $\boldsymbol{\mathcal{U}} = (\cdots, \mathcal{U}_k = M\left(\frac{r_k}{N+1}\right)(u_k^\star), \cdots)^T$ and

$$\boldsymbol{U}(p) = diag(\cdots, e^{-j2\pi pk}, \cdots)_{N_1\le k\le N_2}, \tag{22}$$

$$\left(\boldsymbol{V}^{\Delta,1}(q)\right)_{k,l} = \frac{(cos(\pi(k-l-q)) - sinc(k-l-q))}{k-l-q} \tag{23}$$

$$\left(\boldsymbol{V}^{\Delta,0}(q)\right)_{k,l} = sinc(k-l-q). \tag{24}$$

Moreover the term $\hat{\alpha}$, that represents a consistent estimator for $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\}$ is given by [37]:

$$\hat{\alpha} = \frac{(\sigma_n^\star)^2}{N}\frac{||\widetilde{\Delta}_N(\boldsymbol{\theta}^\star + N^{-1/2}\mathbf{v}^0) - \widetilde{\Delta}_N(\boldsymbol{\theta}^\star)||}{||\mathbf{K}(\boldsymbol{\theta}^\star)\mathbf{v}^0||}, \tag{25}$$

where $\mathbf{v}^0 \sim \mathcal{N}(\mathbf{0}, \varrho\mathbf{I})$ is a "small perturbation" vector. Regarding the *score function* $M(\cdot)$ many choices are possible (see e.g. [29]). However, the one that provide a good trade of between semiparametric efficiency and robustness is the complex *van der Waerden* score function:

$$M_{vdW}(t) \triangleq \sqrt{\Phi_G^{-1}(t)}, \quad t\in(0,1), \tag{26}$$

where Φ_G^{-1} indicates the inverse function of the cdf of a Gamma-distributed random variable with parameters $(1,1)$. Finally, a good choice for the preliminary estimator $(\boldsymbol{\theta}^\star)^\top = [\rho^\star, \Phi^\star, (\boldsymbol{\eta}^\star)^\top]$ and $\sigma_n^\star$ is the Gaussian-based MMLE since, as it was shown in [23], [24], is $\sqrt{N}$-consistent as required. The estimates of the Gaussian-based MMLE $\boldsymbol{\theta}^\star$ are given by:

$$\boldsymbol{\eta}^\star = \arg\max_{\boldsymbol{\eta}}\left\|\boldsymbol{\Pi}_{\boldsymbol{\mu}(\boldsymbol{\eta})}\mathbf{x}\right\|^2 \tag{27}$$

$$\rho^\star = \left|\left[\boldsymbol{\mu}^H(\boldsymbol{\eta}^\star)\boldsymbol{\mu}(\boldsymbol{\eta}^\star)\right]^{-1}\boldsymbol{\mu}^H(\boldsymbol{\eta}^\star)\mathbf{x}\right| \tag{28}$$

$$\Phi^\star = \arg\left\{\left[\boldsymbol{\mu}^H(\boldsymbol{\eta}^\star)\boldsymbol{\mu}(\boldsymbol{\eta}^\star)\right]^{-1}\boldsymbol{\mu}^H(\boldsymbol{\eta}^\star)\mathbf{x}\right\} \tag{29}$$

where $\boldsymbol{\Pi}_{\mathbf{A}} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$ is the orthogonal projector over S, with $S = span(\mathbf{A})$ and $\mathbf{A}$ a matrix. Similarly, as preliminary estimator of the noise variance, we can used the MMLE, which yields to:

$$(\sigma_n^\star)^2 = \|\boldsymbol{x} - \rho^\star e^{j\Phi^\star}\boldsymbol{\mu}(\boldsymbol{\eta}^\star)\|^2/N. \tag{30}$$

IV. M-estimator for $\bar{\boldsymbol{\theta}}$

The M-estimator is a general class of estimators that generalizes the ML estimator. In fact, it is based on the generalization of a function that may be different from the likelihood one. Given a general loss function $\rho(\cdot)$, the M-estimator for a parameter $\bar{\boldsymbol{\theta}}$ is the solution to the following optimization problem:

$$\hat{\boldsymbol{\theta}}_M = \arg\min_{\boldsymbol{\theta}}\sum_{k=1}^{N}\rho(x_k, \boldsymbol{\theta}, \bar{\sigma}_n), \tag{31}$$

In Gaussian-based inference, the score function should ideally be chosen to minimize the expected squared error. In robust statistics, the Huber M-estimator is often used to reduce the influence of outliers by employing a modified loss function. The Huber loss function is defined as:

$$\rho_\delta(v) = \begin{cases} v^2 & \text{if } |v|\le\delta, \\ \delta(2|v|-\delta) & \text{if } |v|>\delta, \end{cases} \tag{32}$$

where $v_k = x_k - f_k(\boldsymbol{\theta})$ is the residual, and δ is a threshold that controls the transition between quadratic and linear behaviour. The Huber estimator minimizes the sum of these losses across all data points:

$$\hat{\boldsymbol{\theta}}_M = \arg\min_{\theta}\sum_{k=1}^{N}\rho_\delta(v_k/\hat{\sigma}_n). \tag{33}$$

where an auxilary estimate $\hat{\sigma}_n$ of the scale $\bar{\sigma}_n$ is required. The parameter δ is chosen based on the target asymptotic relative efficiency (ARE) at a given distribution. Thus, $\delta_{0.95} = 1.345$ indicates that the M-estimator based on Huber's loss function poses an ARE of 0.95 at the standard normal distribution.

[1] The family of score functions is defined in [36, Sect. 2.2], [35, Ch. 13]

A. Iterative Reweighted Least Squares (IRLS)

The M-estimation problem can be solved using the Iteratively Reweighted Least Squares (IRLS) method. The basic idea behind IRLS is to iteratively solve weighted least squares problems by updating the weights at each step. This process is efficient for solving M-estimators with non-quadratic loss functions. The IRLS algorithm for the Huber estimator proceeds as follows:

1) Initialize $\hat{\boldsymbol{\theta}}_M^{(0)}$ and $\hat{\sigma_n}^{(0)\ 2}$. In our simulations, we used the Gaussian-based MMLE estimates (equations (27)-(30)) as they are $\sqrt{N}$-consistent.
2) Iterate until convergence for j:
   a) Update residual $v_k^{(j)} = \left(x_k - f_k(\hat{\boldsymbol{\theta}}_M^{(j)})\right)$.
   b) Update scale $\hat{\sigma}_n^{(j)} = \|\boldsymbol{v}^{(j)}\|/\sqrt{N}$.
   c) Update weight $\boldsymbol{W} = diag\left(\frac{\psi(v_k^{(j)}/\hat{\sigma}_n^{(j)})}{v_k^{(j)}/\hat{\sigma}_n^{(j)}}\right)$.
   d) Compute the weighted least square $\hat{\boldsymbol{\theta}}_M^{(j)} = \arg\min_{\boldsymbol{\theta}} \|\boldsymbol{v}^{(j)}\|_{\boldsymbol{W}}^2 = \arg\min_{\boldsymbol{\theta}} \|\sqrt{\boldsymbol{W}}\boldsymbol{v}^{(j)}\|^2$

with $diag(\cdot)$ operator mapping a vector to the diagonal elements of a matrix and $\psi(v) = \delta\rho(v)/\delta v$ the score function. For the Huber loss function:

$$\psi_\delta(v) = \begin{cases} v, & \text{if } |v| \leq \delta, \\ \delta \cdot \mathrm{sgn}(v), & \text{if } |v| > \delta. \end{cases} \tag{34}$$

and since v is a complex number, $\mathrm{sgn}(v) = v/|v|$, which extracts the "direction" of the complex number while normalizing it to have unit magnitude. Moreover, the solution of the weighted least square for the parameter of interest yields to $\hat{\boldsymbol{\eta}}_M = \arg\max_{\boldsymbol{\eta}} \left\| \boldsymbol{\Pi}_{\sqrt{W}\boldsymbol{\mu}(\boldsymbol{\eta})} \sqrt{W}\mathbf{x} \right\|^2$. Finally, the convergence is reached when $\|\hat{\boldsymbol{\theta}}_M^{(j)} - \hat{\boldsymbol{\theta}}_M^{(j-1)}\| < \epsilon$. However, with a single iteration the algorithm already shows a (please refers to Section V) considerable improvement over the MMLE.

V. Simulation and Discussion

We consider a scenario in which a GPS L1 C/A signal [6] is received by a GNSS receiver. The true signal model assumes that the noise follows a complex-centered t-distribution [26, Sec. 4.6.1.1] with $\upsilon = 1.5$ degrees of freedom (or *shape parameter*), which controls the deviation from Gaussianity, and a scale parameter μ. The second-order modular variate $\mathcal{Q}$ of a t-distribution follows a scaled F-distribution such that $\mathcal{Q} \sim \mu^{-1} F_{2,2\upsilon}$ [32, Sec. IV.A]. To satisfy the condition $E\{\mathcal{Q}\} = 1$, the scale parameter must be set as $\mu = \frac{\upsilon}{\upsilon-1}$. Furthermore, for the t-distribution, it has been shown in [28] that $E\{\mathcal{Q}\bar{\psi}(\mathcal{Q})^2\} = \frac{\mu(\upsilon+1)}{\upsilon+2}$, and the output SNR is given by $SNR_{out} = |\bar{\alpha}|^2 \mathbf{s}^H \mathbf{s}/\bar{\sigma}_n^2$.

Figures 1 and 2 illustrate the root mean square error ($\sqrt{MSE}$) performance of the MMLE, the M-estimator derived in (32) with $\delta_{0.95} = 1.345$ and one iteration, and two R-estimators derived in (17). The first R-estimator uses the MMLE as a preliminary estimator, while the second one uses the output of the M-estimator as its preliminary estimator. The evaluation is conducted for the parameters of interest, $\bar{\boldsymbol{\eta}}^T = [\bar{\tau}, \bar{b}]$, as a function of SNR_{out}. The analysis considers a GNSS receiver operating at a sampling frequency of $F_s = 4$ MHz with an integration time of 1 ms. The results are obtained from 1000 Monte Carlo iterations.

The results demonstrate that the $\sqrt{MSE}$ of the MMLE asymptotically converges to the MCRB, which coincides with the Complex-Gaussian CRB, confirming the findings in [24]. Additionally, we observe that the RMSE of the R-estimator in (17) also asymptotically converges to the SCRB given in (9). The M-estimator converges asymptotically to an intermediate performance level between the MCRB and the SCRB, showing that it is not asymptotically efficient. However, it converges faster than both the MMLE and the R-estimator. Consequently, using the M-estimator as a preliminary estimator for the R-estimator combines desirable properties—faster convergence and asymptotic efficiency. However, this comes at the cost of increased computational complexity.

VI. Conclusion

In this work, we analyzed the performance of time-delay and Doppler estimation in the presence of non-Gaussian noise. Specifically, we considered the M-estimator, and R-estimator, highlighting their theoretical properties and asymptotic behavior. The R-estimator, constructed using rank-based statistics, achieves asymptotic efficiency with respect to the SCRB that for our application, is equal to the classical CRB. Consequently, the R-estimator achieves the optimal parametric performance without any a-priori knowledge of the noise distribution. On the other hand, the M-estimator exhibits an intermediate performance between the MCRB and SCRB, showing that it is not fully efficient but converges faster than the other estimators. By leveraging the M-estimator as a preliminary step for the R-estimator, we achieve a trade-off between faster convergence and asymptotic efficiency at the cost of increased computational complexity.

Acknowledgments

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[2] The normalized median absolute deviation (MAD) can be also used as a scale estimator.

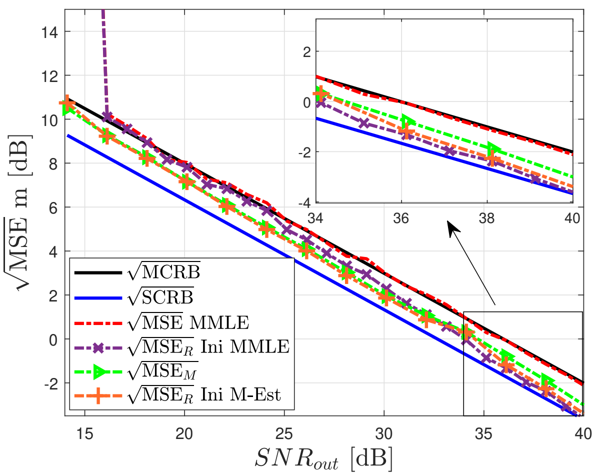

Fig. 1. RMSE of the MMLE M-estimator and R-estimators for the time-delay considering complex centered t-dist. with $\upsilon = \{1.5\}$.

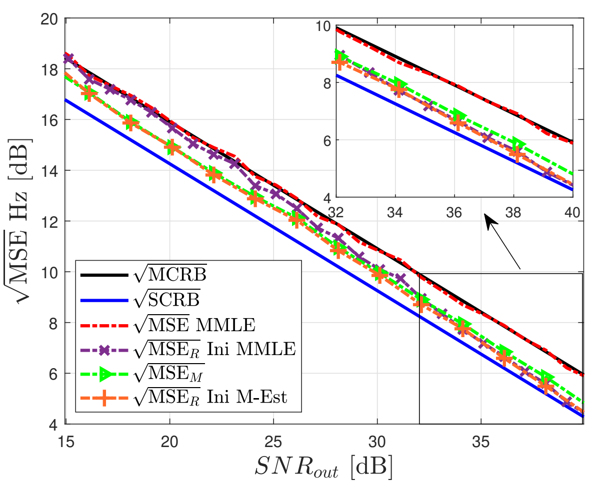

Fig. 2. RMSE of the MMLE M-estimator and R-estimators for the Doppler considering complex centered t-dist. with $\upsilon = \{1.5\}$.

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