Universal Sampling Schemes for Sparse Signal Recovery via Discrete Sine Transform Type-I Even

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***Abstract*—In this work, we address the design of universal sampling patterns, which ensure the recovery of sparse signals from compressed sensing (CS) in a transform domain. Our proposed sampling approach is a generalization of an arithmetic sequence, namely *folded arithmetic sequence* (FAS) of difference d. Considering the Discrete Sine Transform Type-I even (DST1e), we rigorously characterize the differences d that guarantee perfect sparse signal recovery from FAS sampling in the DST1e domain. For such differences d, we prove that the corresponding measurement matrix has maximum spark. Therefore, FAS is a universal sampling scheme for DST1e. Moreover, this approach constitutes the first universal sampling pattern for a Discrete Sine Transform. Simulations illustrate the good behavior of CS solvers when using FAS approach in the DST1e domain.**

***Index Terms*—Compressed sensing, sparse signals, universal sampling pattern, spark, DST.**

I. INTRODUCTION

Discrete Trigonometric Transforms (DTTs) have become an alternative to Discrete Fourier Transform (DFT) in some signal processing applications. For instance, Discrete Cosine Transforms (DCTs) are widely used for signal and image compression [1], and constitute a good alternative in telecommunications, outperforming the DFT in some scenarios [2], [3]. Regarding the Discrete Sine Transforms (DSTs), they are also used in applications such as data compression, adaptive digital filtering, image restoration/interpolation, and graph signal processing [4]–[7]. In particular, the Discrete Sine Transform Type-I even (DST1e) presents important properties: First of all, it outperforms the DCT approach in some scenarios, since DST1e is close to the optimal Karhunen-Loeve transform for first-order Markov stationary signals with low correlation coefficients [1], [7]. Besides, DST1e can be implemented by fast algorithms recently improved [7]. Moreover, DST1e equals its inverse (up to a constant factor) [8], so its inverse transform is straightforward.

In this paper, we address the problem of reconstruction of sparse signals from a small number of measurements in their DST1e transformed vector. From compressed sensing theory [9]–[11], it is possible to recover s-sparse signals by means of a set of $p \geq 2s$ samples in a transform domain, whenever the measurement matrix has maximum spark. In that case, the corresponding sampling pattern is said to be *universal*.

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For a given transform matrix, proving that it presents maximum spark is a very difficult mathematical issue [10]: recall that a matrix with p rows has *maximum spark* [9] if any set of p out of its columns are linearly independent. For the DTTs, in [12] it was shown that all even-type DCTs present maximum spark in its first rows. This result was extended in [13], where a universal arithmetic sampling pattern was designed for the Discrete Cosine Transform Type-I even (DCT1e). Unfortunately, that sampling scheme has not been mathematically proven to be universal for the rest of DTTs.

To our knowledge, there are no results regarding the maximum spark for DSTs in the literature. For this reason, in this work we investigate the design issue of universal sampling schemes for the DST1e. First, we present a new sampling pattern, the *folded arithmetic sequence (FAS)*, based on an arithmetic sequence of difference d, similar -but not equal- to the one proposed in [13]. Then, we derive important necessary and sufficient conditions on d that guarantee maximum spark of the corresponding DST1e measurement submatrix. In this way, we obtain universal FAS patterns for the DST1e transform.

The paper is organized as follows: Section II presents the proposed FAS sampling pattern. Section III provides the main theoretical contributions of this work, say, the theorems that prove the maximum spark of the proposed solution when using the DST1e. Section IV illustrates some numerical simulations of the performance of the proposed approach, and Section V summarizes the main conclusions.

II. DESIGN OF THE FOLDED ARITHMETIC SEQUENCE

In this section we present our new sampling scheme from the components of a vector. For simplicity, we will assume that the vector length is $(N-1)$ and its components are indexed as $m = 1, \ldots, N-1$.

Let us fix two integers $1 \leq d, p \leq N-1$, and consider the set $C = \{dm,\ m = 1, \ldots, p\}$ of p numbers which form an arithmetic sequence of difference d:

$$C = \{d, 2d, \ldots, pd\}. \tag{1}$$

If $pd < N$, these indices correspond directly to p samples of the original vector of length $N-1$. In this case, the approach is well-known: it may be considered as the output of a sampler that works at a fixed rate, as in [14].

However, the condition $pd < N$ is too strong: for high values of the sampling difference d, a very small amount of measurements p could be applied, and only sparse signals of small order $s = p/2$ would be reconstructed.

To overcome this problem, we propose an extension of this sampling scheme for $pd \geq N$, which is valid for any amount of measurements $p \leq (N-1)$, and any difference d of the arithmetic sequence, whenever d and N are coprime. This condition on d guarantees that none of the numbers $d, 2d, ..., pd$ equals a multiple of N, and this is necessary for our procedure, as will be explained.

With this in mind, let us present our method: for any number $dm \in C$, there are two cases:

- If dm lies in an interval of the kind $(2kN, (2k+1)N)$ for an integer k, then dm is replaced by the sample index

$$m' = dm - 2Nk, \quad 1 \leq m' \leq N-1. \tag{2}$$

  Note that this expression is also valid for the numbers $0 < dm < N$ (being $k = 0$), which remain unchanged.
- If dm lies in an interval of the kind $((2k-1)N, 2kN)$ for an integer k, then we replace dm by the sample index

$$m' = 2Nk - dm, \quad 1 \leq m' \leq N-1. \tag{3}$$

In summary, it suffices to substitute the index dm by the sample index m' defined as

$$m' = |2Nk - dm|, \quad 1 \leq m' \leq N-1. \tag{4}$$

Thus, Eq. (4) yields the general procedure for our novel sampling pattern, which can be applied to any $m = 1 \ldots, p$.

The following important result gives the condition that guarantees that the obtained indices are all different, so this way we get the desired number p of samples:

Proposition: If d and $2N$ are coprime, then the proposed procedure yields exactly p different sampling indices.

Proof: Let us suppose that there exist two different numbers $dm_1 \neq dm_2 \in C$ which provide the same index m' in Eq. (4). This happens if and only if

$$2Nk_1 - dm_1 = \pm (2Nk_2 - dm_2)$$
$$\iff 2N(k_1 \mp k_2) = d(m_1 \mp m_2)$$

for some integers k_1, k_2 . But this is not possible since d and $2N$ are coprime, and $0 < |m_1 \mp m_2| < 2p - 1 < 2N$. Hence, the p indices obtained are all different. □

Summary: From the set C of Eq (1), we build the set $I \subseteq \{1, \ldots, N-1\}$ which contains the new sampling indices defined by Eq.(4). If d is coprime to $2N$, then I is the proposed sampling pattern formed by p different indices. It is easy to see that they result from folding the elements of the arithmetic sequence C with respect to N or 0. For this reason, we will call the proposed sampling scheme ***folded arithmetic sequence***.

Remark 1: The proposed sampling pattern provides a set of indices I which form arithmetic sequences of the same difference d . Thus, they can be considered as the output of a set of asynchronous samplers, all of them with the same fixed rate. In this sense, this approach is similar (but not equal) to the ones summarized in [14] for the DFT, and the one presented in [13] for the DCT1e, as will be seen in the next example.

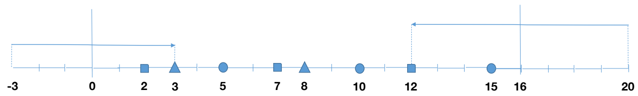

Fig. 1. Example of the proposed folded arithmetic sequence for $N = 16$, $p = 8$ and $d = 5$, as explained in Section II.A.

A. Example

For a 15-length signal, then $N = 16$, so we can choose any difference d coprime with N, say, any odd number $1 \leq d \leq 15$. By taking $d = 5$, let us show how to design $p = 8$ samples with our FAS approach. The first eight multiples of d are $\{5, 10, 15, 20, 25, 30, 35, 40\}$. Let us apply our method:

1) First, We keep the samples in $[1, N-1]$, say, $\{5, 10, 15\}$.
2) Secondly, the numbers lying in the interval $(N, 2N) = (16, 32)$ are $\{20, 25, 30\}$; following Eq. (3) each number $32 - k$ should be replaced by the index k, so the respective new sample indices are $\{12, 7, 2\}$. (Notice that they also form an arithmetic sequence of difference $-d = -5$ whose next element would be -3).
3) Finally, the numbers lying in $(2N, 3N) = (32, 48)$ are simply replaced by their remainders modulo $(2N = 32)$ as in Eq. (2). Hence, the numbers $\{35, 40\}$ yield the new sample indices $\{3, 8\}$ (with difference $d = 5$, as expected).

Fig. 1 shows a diagram for this example: notice that this procedure is equivalent to folding the numbers of the arithmetic sequence on the right of $[0, 16]$ with respect to $N = 16$ (marked with squares), continue to the left with the arithmetic sequence of difference $-d$, and then folding the resulting negative numbers with respect to 0 (marked with triangles). Therefore, this is a folded arithmetic sequence.

Finally, we have obtained the sampling set

$$I = \{5, 10, 15\} \cup \{2, 7, 12\} \cup \{3, 8\}$$

which corresponds to three arithmetic sequences of the same difference $d = 5$.

Remark 2: This FAS pattern is different from the one proposed in [13] for DCT1e: in that procedure, from a L-length vector with components indexed as $1 \leq m \leq L$, the first component is always chosen (indexed as $m = 1$), whereas our FAS scheme starts from $m = d$. Besides, in that approach, $N = L-1$ instead of $N = L+1$. Finally, the example given in [13] with the same setting (vector length 15, difference $d = 5$ and $p = 8$ samples), yielded the sampling pattern indexed as $\{1, 6, 11\} \cup \{4, 9, 14\} \cup \{3, 8\}$, which is obviously different from our FAS proposed pattern I. Therefore, although the method given in [13] may be considered a folded arithmetic sequence, it is clear that it provides a solution different from our proposed FAS scheme.

III. Perfect recovery from the DST1e domain

Once the FAS sampling scheme has been introduced, we now analyze its validity for perfect sparse reconstruction in the DST1e domain, Recall that the DST1e matrix of order $N-1$, $\mathbf{S}_{1e}$, is defined in [8] as

$$[\mathbf{S}_{1e}]_{k,n} = \sin\left(\frac{kn\pi}{N}\right),\ 1 \leq k, n \leq N-1.$$

We must study the spark of submatrices of $\mathbf{S}_{1e}$ built by p rows indexed as multiples of an integer $1 \leq d < N$ say, $k = dm$, $m = 1, ..., p$. Let $\mathbf{A}$ denote the matrix whose entries are

$$a_{m,n} = \sin\left(\frac{\pi dmn}{N}\right)\ m = 1, ..., p,\ n = 1, \dots, N-1. \quad (5)$$

Notice that, if $\mathbf{A}$ is a submatrix of $\mathbf{S}_{1e}$, then necessarily none of the numbers $\{d, 2d, \dots, pd\}$ is a multiple of N. In effect, if there exists any $1 \leq m \leq p$ such that $md = NL$ for an integer L, then the row of index dm would be null, since

$$\sin\left(\frac{\pi mdn}{N}\right) = \sin(\pi Ln) = 0.$$

But this row would not correspond to any row of the DST1e matrix, because $\mathbf{S}_{1e}$ is invertible.

Remark 3: We have just shown that a matrix $\mathbf{A}$ of the form provided in Eq.(5) is a submatrix of $\mathbf{S}_{1e}$ only if the set $C = \{d, 2d, \dots, pd\}$ does not contain multiples of N. From now on, we will assume this condition.

Remark 4: Luckily, for the definition of the matrix $\mathbf{A}$ it is not necessary to assume $dp < N$ either: in case $dm > N$, we can use the expressions given in Section II, since Remark 3 assures that dm is not a multiple of N. Thus, we easily write $dm = 2Nk \pm m'$ as in Eq. (4) for some $0 < m' < N$: the m-th row of matrix $\mathbf{A}$ of Eq. (5) has entries

$$\sin\left(\frac{(dm)\,n\pi}{N}\right) = \sin\left(\frac{(2Nk \pm m')\,n\pi}{N}\right) = \pm\sin\left(\frac{m'n\pi}{N}\right)$$

so this row equals the m'-th row of $\mathbf{S}_{1e}$, or its opposite. Fortunately, changing the sign of a complete row of $\mathbf{A}$ does not affect its spark, so we conclude that we can consider that the rows of $\mathbf{A}$ correspond to the rows of matrix $\mathbf{S}_{1e}$ indexed by the set I designed in Section II. In other words, we can consider that $\mathbf{A}$ is a submatrix of $\mathbf{S}_{1e}$.

Our aim is to find the characterization of d which yield maximum spark of $\mathbf{A}$. The following result provides it, and it constitutes one of the main contributions of this work:

Theorem: *For any integers $1 \leq d, p < N$, the $p \times (N-1)$ matrix $\mathbf{A}$ defined in Eq.(5) has maximum spark (p) if and only if d and $2N$ are coprime.*

Proof: Let us first prove that the coprimality of d and $2N$ is a necessary condition for the maximum spark of $\mathbf{A}$. It suffices to show that if d is even or d is not coprime to N, then $\mathbf{A}$ has not maximum spark. If d is even, we note that the first column of $\mathbf{A}$ $(n = 1)$ and its last column $(n = N-1)$ have opposite components:

$$\begin{aligned} a_{m,N-1} &= \sin\left(\tfrac{\pi dm(N-1)}{N}\right) = \sin\left(dm\pi - \tfrac{\pi dm}{N}\right) \\ &= -(-1)^{dm}\sin\left(\tfrac{\pi dm}{N}\right) = -a_{m,1}. \end{aligned}$$

Therefore, if d is even, these two columns of $\mathbf{A}$ are proportional, hence linearly dependent, so the spark of $\mathbf{A}$ is not maximum. Here we had assumed that $dp < N$; if $dm > N$, the proposed sampling scheme will substitute the even number dm by the index m' defined by Eq. (4) which will be also even. All the corresponding rows are even-indexed, so this scheme will also provide a submatrix whose first and last columns are proportional; therefore, the spark will not be maximum either.

Let us now suppose that d,N are not coprime, there exist integers $1 < L < N$ and $1 \leq k < d$ such that $dL = Nk$. Hence the column of index $L < N$ has entries $\sin(\pi mdL/N) = \sin(\pi mk) = 0$. In other words, the $L-$th column is null, so the matrix $\mathbf{A}$ obviously does not have maximum spark. This concludes the first part of the proof.

Now we prove the second part of the theorem, say, if d is coprime to $2N$, then $\mathbf{A}$ has maximum spark, i.e., any set of p of its columns are linearly independent. Let us consider any p columns with indices $0 < n_1 < n_2 < ... < n_p < N$. We build the $p \times p$ square submatrix $\mathbf{B}$ formed by these p generic columns; its entries are

$$b_{m,n} = \sin\left(\frac{\pi mnd}{N}\right),\ 1 \leq m \leq p,\ n \in \{n_1, n_2, ..., n_p\}.$$

To prove that $\mathbf{B}$ is invertible, it suffices to demonstrate that the unique row vector $\mathbf{a} = (a_1, a_2, ..., a_p)$ such that $\mathbf{aB} = \mathbf{0}$ is $\mathbf{a} = \mathbf{0}$. Let us rewrite the condition $\mathbf{aB} = \mathbf{0}$ as

$$\sum_{m=1}^{p} a_m \sin\left(\frac{\pi mdn_k}{N}\right) = 0, \quad k = 1, ..., p. \qquad (6)$$

By using the complex unitary numbers

$$w_k = \exp\left(\frac{\pi dn_k}{N} j\right) \quad k = 1, \dots, p \qquad (7)$$

then we can express $w_k^m - w_k^{-m} = 2j\sin\left(\frac{\pi mdn_k}{N}\right)$ so Eq. (6) is easily rewritten as

$$\sum_{m=1}^{p} a_m \left(w_k^m - w_k^{-m}\right) = 0, \quad k = 1, ..., p.$$

If we multiply the latter expression by w_k^p, we observe that each w_k of Eq. (7) is a root of the polynomial

$$q(z) = \sum_{m=1}^{p} a_m \left(z^{m+p} - z^{p-m}\right) \qquad (8)$$

of degree $2p$. Note that the coefficients of q are antisymmetric with respect to the central ones; hence, for any $z \neq 0$, $q\left(z^{-1}\right) = -q(z)z^{-2p}$. This implies that for any root w_k, also w_k^{-1} is also a root of q :

$$w_k^{-1} = \exp\left(-\frac{\pi n_k d}{N} j\right) \quad k = 1, ..., p. \qquad (9)$$

Let us count how many different roots of q there are: it suffices to see that their arguments do not differ in an integer multiplied by 2π. If there exist $0 < n_k < n_{k'} < N$ such that

$$\frac{\pi n_k d}{N} \pm \frac{\pi n_{k'} d}{N} = 2\pi m \iff (n_k \pm n_{k'})\, d = 2mN$$

then d would not be coprime to $2N$, and it is impossible. Therefore, q has at least $2p$ different roots: $\{w_k,\ w_k^{-1},\ k = 1, \ldots, p\}$. Note that they are not real numbers: from Eqs. (7) and (9) , their arguments cannot be equal to 0 or π since d is coprime to $2N$. Finally, it suffices to notice that $z = 1$ is a real root of the polynomial q defined in Eq. (8) . Hence, q has $2p + 1$ different roots, more than its degree, so q should be the null polynomial, and the claim follows. $\square$

Corollary: For any p, the first p rows of the DST1e matrix have maximum spark.

This corollary is straightforward since $d = 1$ is always coprime to any integer $2N$. As a consequence, the first consecutive samples of a signal $\{1, 2, \cdots, p\}$ constitutes a universal sampling pattern for DST1e. Finally, the FAS example given in Section II.A verifies that d and $2N$ are coprime, so it is also a universal scheme for DST1e.

IV. Experimental results

In this section we show some simulations where traditional compressed sensing (CS) solvers have been applied to the proposed universal FAS sampling pattern in the DST1e domain. In each experiment, first we set the parameters N (the length of the signal plus 1), p (number of measurements in the transform domain), and the difference parameter d (coprime to $2N$). Then, for each sparsity value s, $1 \leq s \leq p$, a s-sparse signal $\mathbf{x}$ of length $N - 1$ is built: its s nonzero locations are drawn at random, and its respective nonzero values are drawn from a normal Gaussian distribution $\mathcal{N}(0, 1)$. We compute its DST1e transform vector $\mathbf{b} = \mathbf{S}_{1e} \cdot \mathbf{x}$, and extract its p components indexed by the folded arithmetic sequence of difference d.

Secondly, from these p measurements we apply CS solvers, such as: basis pursuit (BP), smoothed ℓ_0 algorithm (SL0) [15], and Orthogonal Matching Pursuit in its modified version (OMP1) [16]. Each of these algorithms compute an estimation of the sparse signal $\mathbf{x}$. Finally, the experiment is repeated 100 times for each sparsity value s, and the empirical recovery rate of each algorithm is computed for each sparsity value s.

Let us show some results obtained for $N = 16$, $p = 8$: with this setting, we can choose any value of difference d coprime to $2N = 32$, because they guarantee universal patterns for DST1e. On the one hand, $d = 1$ yields the first $p = 8$ consecutive samples of the transformed vector $\mathbf{b}$. On the other hand, $d = 5$ provides the FAS universal sampling pattern in $\mathbf{b}$ already designed in Section II.A, and depicted in Fig. 1: it corresponds to the samples $\{2, 3, 5, 7, 8, 10, 12, 15\}$.

Fig. 2 shows the corresponding recovery rates for the CS solvers (SL0, BP, OMP1) along the sparsity values s, with our new FAS approach with $d = 5$. As expected, high recovery rates are obtained through these algorithms for sparsity values $s \leq p/2 = 4$. Recall that CS theory never ensures recovery for sparsity values $s > p/2 = 4$. Similar good results have been also achieved -but not shown here- for the case $d = 1$.

For higher values of N, we have also analyzed the performance for all possible values of d: by taking $N = 64$, $p = 32$, we have considered all the differences $d < N$ coprime to $2N = 128$, say, all odd numbers $1 \leq d \leq 63$. The OMP1 recovery rates with DST1e are displayed at the top of Fig. 3, where each line corresponds to a different value of d. We observe very good behaviour for all differences d (including the worst case, $d = 1$); moreover, for many choices of d (for instance, $d = 11$ or $d = 53$), the performance of the CS solver is highly improved.

In order to compare the behaviour of DST1e with DCT1e, simulations have been carried out using the arithmetic approach presented in [13] for DCT1e, with exactly the same setting: $N = 64$, $p = 32$ and odd differences $1 \leq d \leq 63$. The resulting OMP1 recovery rates with DCT1e, illustrated at the bottom of Fig. 3, present very similar behaviour for all values d; but the key point is that they are clearly outperformed by the proposed DST1e approach at the top of Fig. 3.

Finally, we have also compared simulations with random s-sparse signals with concentrated support, say, whose support is an (unknown) interval of length s. In [13] it was observed that, in this scenario, the performance depended greatly on the value of d for DCT1e. We have also analyzed it for the DST1e: for the same setting $N = 64$ and $p = 32$, Fig. 4 compares the recovery rates of the BP solver for $d = 1$ and $d = 13$ (coprime to $2N$) for sparse signals of concentrated support: both for DST1e and DCT1e, the choice of $d = 13$ outperforms greatly the recovery rate of $d = 1$, which decreases dramatically.

We conclude that the behaviour of the proposed approach for DST1e equals the DCT1e for signals with concentrated support, but in the general sparse case, the proposed DST1e scheme outperforms clearly the DCT1e approach for most values of difference d. Hence, further studies should be carried out to obtain the best choice of d for any given values N and p, and for any trigonometric transform.

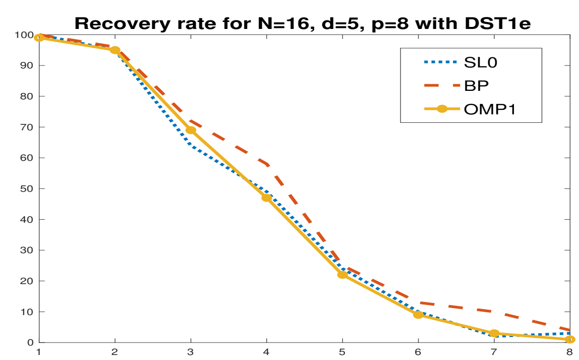

Fig. 2. Recovery rate versus sparsity value s for the CS solvers SL0, BP and OMP1 for the DST1e. In all cases $N = 16$, and $p = 8$ samples are selected from the DST1e vector, following the FAS proposed sampling pattern of difference $d = 5$.

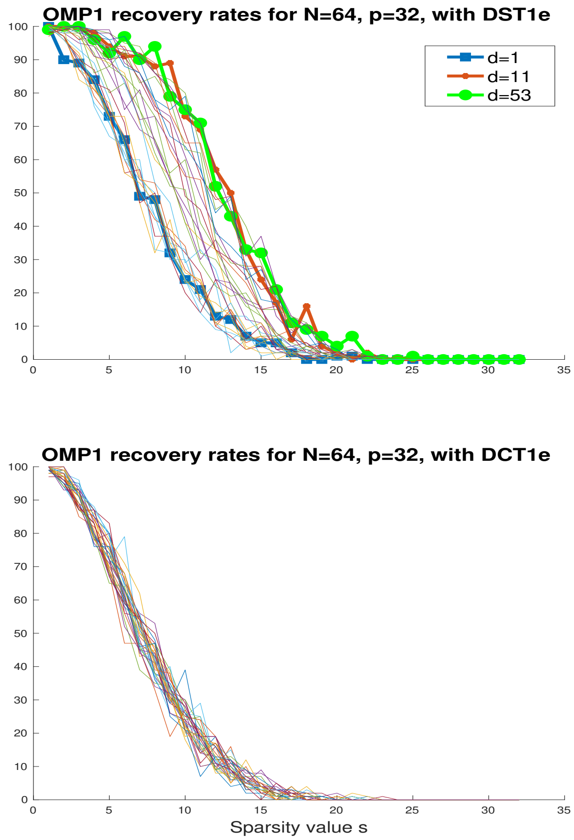

Fig. 3. Comparison of OMP1 recovery rates of sparse signals, versus sparsity value, for $N = 64$, $p = 32$, considering all possible odd differences $1 \leq d \leq 63$, by using the FAS proposed sampling pattern in the DST1e domain (top), and the existing arithmetic approach [13] for DCT1e (bottom).

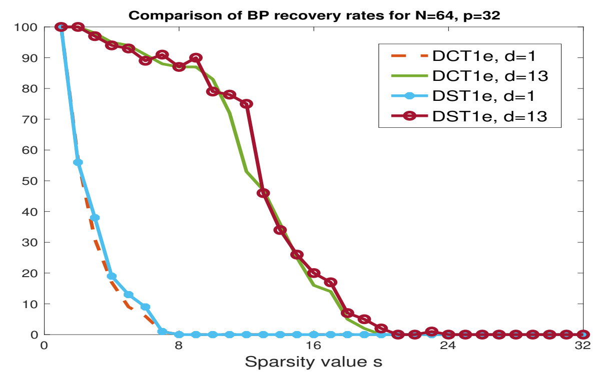

Fig. 4. Comparison of BP recovery rates of sparse signals with concentrated support for $N = 64$, $p = 32$, obtained with differences $d = 1, 13$ when using the DCT1e approach [13] and the proposed FAS scheme for DST1e.

V. Conclusions

In this work, novel universal sampling patterns have been designed. They consist of a family of folded arithmetic sequences (FAS), that can be obtained by sampling the transform vector at a fixed rate. Considering the DST1e, we demonstrate the necessary and sufficient condition that guarantees that the corresponding measurement matrix has maximum spark, so recovery of sparse signals can be ensured. Such condition is that the difference d of the arithmetic sequence is odd and coprime to the length of the signal plus 1. This important theorem constitutes the first theoretical result on maximum spark for the DSTs. Thus, we have provided a new deterministic universal compressed sensing scheme in the DST1e transform domain. Simulations illustrate the good behaviour of the proposed FAS technique for such values of the difference d with DST1e, greatly improving the existing DCT1e approach. For sparse signals of concentrated support, some values of $d > 1$ outperform the case $d = 1$ of consecutive samples. Future research will address the problem of finding the optimal values of d for each scenario. Further theoretical results should also be developed, in order to show the validity of the proposed method for other trigonometric transforms.

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